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HYDRODYNAMIC RESISTANCE OF AN ARBITRARY PARTICLE TRANSLATING AND ROTATING NEAR A FLUID INTERFACE

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Abstract—A general and systematic procedure is developed for calculating the hydrodynamic force and torque experienced by an arbitrarily-sized, -shaped and -oriented particle undergoing an arbitrarily-directed translational and rotational motion inside one of two semi-infinite immiscible fluids separated by a planar interface. The procedure is developed for the case where the ratio, K, of particle characteristic size, $\mathbf{a}$, to the particle's characteristic distance, d, from the interface is much smaller than unity (i.e. $K \ll 1$). Situations in which the far fields in each of the two fluids are arbitrary Stokes flow fields are also included in our analysis. Expressions derived for force and torque are in the form of a power series in the ratio K. It is demonstrated that the general results presented here can be easily used to derive explicit expressions for force and torque on any given particle in terms of the fluid and flow properties, as well as certain geometrical properties of the particle, provided the solution to a particle-dependent Fredholm-type surface integral equation is known or obtainable.

The utility of the general results in calculating the hydrodynamic resistance of particles is illustrated by the example of an arbitrarily-oriented ellipsoid translating and rotating in a quiescent two-phase fluid. Applications to bodies, such as slender bodies, for which only an approximate solution to the integral equation is available, are also briefly discussed.

1. INTRODUCTION

A particle moving in the vicinity of an interface between two immiscible fluids experiences a force and torque which, depending on the ratio of viscosities of the two fluids, the interface shape, particle geometry and direction of particle motion, may be higher or lower than those experienced by the same particle in an unbounded flow. Apart from depending on the aforementioned factors, the magnitude of the "extra" force and torque also depends on a characteristic particle size, $\mathbf{a}$, and orientation, as well as on the ratio ($K =$) $\mathbf{a}/d$ where d is a characteristic distance of the particle from the interface. Calculation of this type of boundary effect is essential to the understanding of many phenomena of physical and engineering interest. Included among these phenomena are sedimentation, motion of micro-organisms, viscometry (Brenner 1964a), Brownian motion in colloids, lateral migration and drop or bubble flotation, to name a few.

In the recent and not-so-recent past, many research activities have been directed at calculating the hydrodynamic resistances of spherical and other geometrically related particles translating or rotating parallel or normal to a planar fluid–fluid or fluid–solid interface. Exact solutions for this class of problems have been obtained (cf. Brenner 1961; Dean & O'Neill 1963; Kunesh 1971; Schneider *et al.* 1973; O'Neill & Ranger 1979; Lee & Leal 1980) using the eigenfunction method originated by Jeffery (1912, 1915). Other exact solutions obtained by methods other than Jeffery's eigenfunction method include those for a circular disc straddling an interface (Ranger 1978) and an elliptic disc straddling an interface (Falade 1982).

For $K \ll 1$, approximate expressions in the form of a power series in K for the force and torque on a particle moving near a plane interface may be obtained by employing a regular perturbation technique. In this connection, Brenner (1964a) used the flow field of a rotlet singularity oriented normal to a free surface to calculate, to order K^8, the torque on an axisymmetric body rotating near a free surface. Later, Lee *et al.* (1979) extended Lorentz's (1896) theorem for fluid motion in the presence of a plane wall to the general case of a fluid–fluid interface (see also Aderogba & Blake 1978) and used the results to obtain asymptotic expressions for the resistance of a sphere translating and rotating near a fluid interface. Lee & Leal (1980) made comparisons between the asymptotic and "exact" values of force and torque on a sphere. Their general finding was that agreement between the two sets of results is good for $K^{-1} \geqslant 1.4$ except in the special cases where the interface is a solid surface and the particle velocity vector has a non-zero component normal to the surface. In the latter cases differences between the exact and asymptotic values of force and torque become significant for $K^{-1} \geqslant 2.0$. The extended method of Lorentz has also been employed to calculate the force and torque on a slender cylinder translating near a planar interface (Fulford & Blake 1983; Yang & Leal 1983, 1984).

In this paper, a procedure is given for calculating the force and torque on an arbitrarily-sized, -shaped and -oriented particle translating and rotating near a planar interface between two immiscible fluids. For our analysis to be valid, however, the size and location of the particle relative to the interface must be such that $K \ll 1$. Our analysis also allows for the case where, in the absence of the particle, the two fluids are themselves undergoing arbitrary Stokes motion. It is assumed that, in addition to satisfying Stokes equations, both the undisturbed and disturbance fields satisfy the condition of continuity of velocity and tangential stresses across the interface as well as the condition of zero normal velocity at the interface. It is further assumed that the discontinuity in normal stress across the interface does not cause any significant deformation of the planar interface. As shown by Lee *et al.* (1979), the latter assumption is reasonable if either surface tension or gravity forces are much greater than viscous forces (i.e. $U\mu^1/\sigma \ll 1$ or $g\mathbf{a}^2\Delta\rho/\mu^1 U \gg 1$, where σ = interfacial tension, μ^1 = viscosity of fluid I (see figure 1) g = acceleration due to gravity, $\Delta\rho$ = density difference between the two fluids and U is a characteristic flow velocity) or alternatively, if $K \ll 1$. The latter condition has already been assumed in our analysis.

The method used in the development of our general results is the same as the singular perturbation method used by Cox & Brenner (1967) to derive general expressions for the effect of a solid wall of finite extent on the Stokes resistance of an arbitrary particle. In the problem under consideration here, however, the boundary is an interface of a known shape, and therefore, our results are of a less general nature than theirs. Use is also made of the two-phase Stokeslet solution given by Aderogba & Blake (1978) and Lee *et al.* (1979).

The equations governing the problem of an arbitrary particle moving slowly in the vicinity of a planar interface are given in section 2. In section 3, the singular perturbation procedure for calculating the force and torque to any desired order in K is described. Some special cases which afford a reduction in the complexity of the general results of section 3 are discussed in section 4. The general results of section 3 can be used to derive explicit expressions for the force and torque on a given particle if the solution of a particle-dependent Fredholm-type surface integral equation is known or obtainable. To illustrate the steps involved in the passage from general results to particular results, we give in section 5 the solution for an arbitrarily-oriented triaxial ellipsoid translating or rotating near the planar interface between two quiescent fluids. The results in section 3 can also be applied to bodies for which only an approximate solution to the integral equation is available. This fact is demonstrated in section 6 by the example of an arbitrarily-oriented slender circular cylinder translating normal to an interface.

2. GOVERNING EQUATIONS

Consider an arbitrary particle B of characteristic linear dimension, $\mathbf{a}$, translating and rotating with linear and angular velocities V_i' and Ω_i', respectively, inside one of two semi-infinite immiscible fluids (fluid I and fluid II). As in figure 1, let the interface between the two fluids be the plane $x_3' = 0$ relative to a cartesian coordinate system (x_1', x_2', x_3') with origin Q lying inside the plane of the interface. Without loss of generality let B be located in fluid I such that a point O affixed to B has the coordinates $(0, 0, d)$ $(d > 0)$ in the (x_1', x_2', x_3') system. It is presumed that, in the absence of B, there would be Stokes flow

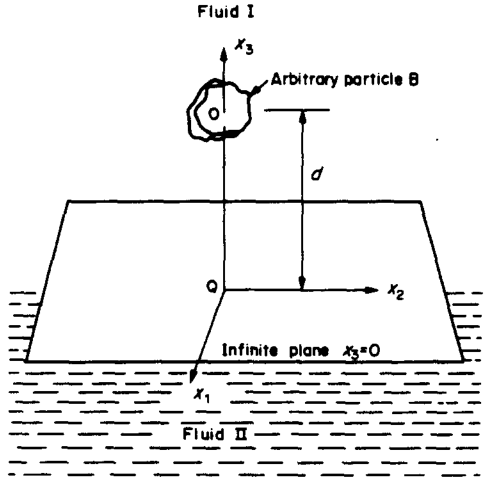

Figure 1. A schematic sketch of the positions of point O, affixed to particle B and the interface in the cartesian coordinate system (x_1', x_2', x_3').

fields $\bar{V}_i^{\mathrm{I}}(x_1', x_2', x_3')$ and $\bar{V}_i^{\mathrm{II}}(x_1', x_2', x_3')$ in the regions $x_3' < 0$ and $x_3' > 0$, respectively. (Note that in this paper superscripts I and II are used, wherever necessary, to distinguish between quantities in fluids I and II, respectively.) Denote by v_i' and p', respectively, the resultant velocity and pressure fields in the fluids. In terms of characteristic fluid speed U, fluid viscosity μ^{I}, and $\mathbf{a}$, the dimensionless quantities V_i, $\bar{V}_i$, Ω_i, v_i, p and x_i may be defined thus:

$$\bar{V}_i = \frac{\bar{V}_i}{U};\quad V_i = \frac{V_i'}{U};\quad \Omega_i = \frac{\Omega'\mathbf{a}}{U};\quad v_i = \frac{v_i'}{U};\quad p = \frac{p'\mathbf{a}}{\mu^{\mathrm{I}}U};\quad \text{and}\quad x_i = \frac{x_i'}{\mathbf{a}}.$$

If the fluids are incompressible, and Reynolds numbers based on U and $\mathbf{a}$ in both fluids are small enough to justify a neglect of inertial terms in the Navier–Stoke's equations, the equations satisfied by v_i and p in both fluids are

$$v_{i,jj}^{\mathrm{I}} - p_{,i}^{\mathrm{I}} = 0,\quad v_{i,jj}^{\mathrm{II}} - \lambda p_{,i}^{\mathrm{II}} = 0, \qquad [1a,b]$$

and

$$v_{j,j} = 0, \qquad [2]$$

where $\lambda = \mu^{\mathrm{II}}/\mu^{\mathrm{I}}$.

In [1a,b] and [2] and throughout this paper, unless the contrary is explicitly stated, Einstein's summation convention is implied when subscripts are repeated. Also, predecession of a subscript by a comma denotes differentiation with respect to the independent variable corresponding to the subscript, i.e.

$$u_{,i} \equiv \frac{\partial u}{\partial x_i}.$$

In addition to [1a,b] and [2], v_i and p are required to satisfy the following boundary conditions:

$$v_i^{\mathrm{I}}(x_1, x_2, 0^+) = v_i^{\mathrm{II}}(x_1, x_2, 0^-),\quad i = 1, 2, 3; \qquad [3a]$$

$$v_3^{\mathrm{I}}(x_1, x_2, 0^+) = v_3^{\mathrm{II}}(x_1, x_2, 0^-) = 0; \qquad [3b]$$

$$\sigma_{3j}^{\mathrm{I}}(x_1, x_2, 0^+) = \lambda\sigma_{3j}^{\mathrm{II}}(x_1, x_2, 0^-),\quad j = 1, 2; \qquad [3c]$$

$$v_i = v_i + \epsilon_{ijk}\Omega_j x_k \text{ on the surface of B}; \qquad [4]$$

and

$$v_i \to \bar{V}_i \quad \text{as} \quad |r| \to \infty. \qquad [5]$$

In [3c], σ_{ij} is the dimensionless stress tensor which is related to v_i and p by

$$\sigma_{ij} = -p\delta_{ij} + v_{i,j} + v_{j,i}.$$

In [4], ϵ_{ijk} is the alternating unit tensor while, in [5],

$$|r| = [x_1^2 + x_2^2 + (x_3 - K^{-1})^2]^{1/2} \quad \text{and} \quad K = \frac{\mathbf{a}}{d}.$$

Let u_i be a disturbance velocity field defined by

$$u_i^{\mathrm{I}} = v_i^{\mathrm{I}} - \bar{V}_i^{\mathrm{I}} \quad \text{and} \quad u_i^{\mathrm{II}} = v_i^{\mathrm{II}} - \bar{V}_i^{\mathrm{II}}, \tag{6}$$

and q be the Stokes pressure field associated with u_i. Since $\bar{V}_i$ satisfies [1a, b], [2], [3a–c] and [5], the equations satisfied by the field (u_i, q) must be:

$$u_{i,jj}^{\mathrm{I}} - q_{,i}^{\mathrm{I}} = 0, \; u_{i,jj}^{\mathrm{II}} - \lambda q_{,i}^{\mathrm{II}} = 0; \tag{7a, b}$$

$$u_{j,j} = 0; \tag{7c}$$

$$u_i^{\mathrm{I}}(x_1, x_2, 0^+) = u_i^{\mathrm{II}}(x_1, x_2, 0^-), \quad i = 1, 2, 3; \tag{8a}$$

$$u_3^{\mathrm{I}}(x_1, x_2, 0^+) = u_3^{\mathrm{II}}(x_1, x_2, 0^-) = 0; \tag{8b}$$

$$\bar{\sigma}_{3j}^{\mathrm{I}}(x_1, x_2, 0^+) = \lambda\bar{\sigma}_{3j}^{\mathrm{II}}(x_1, x_2, 0^-), \quad j = 1, 2; \tag{8c}$$

$$u_i \to 0 \quad \text{as} \quad |r| \to \infty; \tag{9}$$

and

$$u_i = v_i + \epsilon_{ijk}\Omega_j x_k - \bar{V}_i \text{ on the surface of B.} \tag{10}$$

In [8c],

$$\bar{\sigma}_{ij} = -q\delta_{ij} + u_{i,j} + u_{j,i}.$$

If the undisturbed velocity field $\bar{V}_i(x_1, x_2, x_3)$ has no singularities in the neighbourhood of B, then it admits of a Taylor-like power series expansion in this neighbourhood. Thus, without loss of generality, [10] may be replaced by the condition

$$u_i = \alpha_i + \alpha_{ij}x_j + \alpha_{ijk}x_k x_j + \cdots, \quad \text{on B}, \tag{11}$$

where α_i, α_{ij} and α_{jki} etc. are a constant vector, matrix and tensor, respectively. It is also to be noted that if the field $\bar{V}_i$ has no singularity in the region occupied by B, the particle experiences the same resistance in the field (u_i, q) as it does in the field (v_i, p).

In the next section, asymptotic solutions to [7a–c]–[9] and [11] are developed using a singular perturbation technique similar to that used by Cox & Brenner (1967).

3. DERIVATION OF GENERAL EXPRESSIONS FOR FORCE AND TORQUE

To solve [7a–c]–[9] asymptotically, we define inner and outer fields. We postulate that the inner field has the asymptotic expansion

$$u_i^{\mathrm{I}} = {}_0u_i^{\mathrm{I}} + {}_1u_i^{\mathrm{I}} + {}_2u_i^{\mathrm{I}} + \cdots \tag{12a}$$

and

$$q^{\mathrm{I}} = {}_0q^{\mathrm{I}} + {}_1q^{\mathrm{I}} + {}_2q^{\mathrm{I}} + \cdots \tag{12b}$$

Each pair of $({}_nu_i^{\mathrm{I}}, {}_nq^{\mathrm{I}})$ in [12a, b] satisfies [7a–c] and [11] but not [8a–c] and [9].

The outer field, on the other hand, has the asymptotic representation

$$u_i = {}_1\tilde{u}_i + {}_2\tilde{u}_i + \cdots \tag{13a}$$

and

$$q = {}_1\tilde{q} + {}_2\tilde{q} + \cdots \tag{13b}$$

for both fluids I and II. Next we define an outer independent variable $\tilde{x}_i$ by the relation,

$$\tilde{x}_i = Kx_i.$$

In the coordinate system define by $\tilde{x}_i$, each pair of terms $({}_n\tilde{u}_i, {}_n\tilde{q})$ in the outer field is made to satisfy

$${}_n\tilde{u}^{\mathrm{I}}_{i,jj} - K^{-1}{}_n\tilde{q}^{\mathrm{I}}_{,i} = 0, \; {}_n\tilde{u}^{\mathrm{II}}_{i,jj} - \lambda K^{-1}{}_n\tilde{q}^{\mathrm{II}}_{,i} = 0, \tag{14a, b}$$

$${}_n\tilde{u}_{j,j} = 0, \tag{14c}$$

$${}_n\tilde{u}^{\mathrm{I}}_i(\tilde{x}_1, \tilde{x}_2, 0^+) = {}_n\tilde{u}^{\mathrm{II}}_i(\tilde{x}_1, \tilde{x}_2, 0^-), \quad i = 1, 2, 3, \tag{15a}$$

$${}_n\tilde{u}^{\mathrm{I}}_3(\tilde{x}_1, \tilde{x}_2, 0^+) = {}_n\tilde{u}^{\mathrm{II}}_3(\tilde{x}_1, \tilde{x}_2, 0^-) = 0, \tag{15b}$$

$${}_n\tilde{\sigma}_{3j}(\tilde{x}_1, \tilde{x}_2, 0^+) = \lambda\, {}_n\tilde{\sigma}^{\mathrm{II}}_{3j}(\tilde{x}_1, \tilde{x}_2, 0^-), \quad j = 1, 2 \tag{15c}$$

and

$${}_n\tilde{u}_i \to 0 \quad \text{as} \quad |\tilde{r}| \to \infty, \quad \tilde{r} = Kr. \tag{16}$$

The outer interface condition satisfied by the inner field and the inner boundary condition satisfied by the outer field are to be obtained by making both fields satisfy the asymptotic matching conditions.

3.1. *Zero-order inner field*

The zero-order inner field in fluid I is determined uniquely by making it satisfy the unbounded flow outer condition,

$${}_0u^{\mathrm{I}}_i \to 0 \quad \text{as} \quad |r| \to \infty, \tag{17}$$

in addition to satisfying [7a–c] and [11]. It can be shown (Bilby *et al.* 1975) that

$${}_0u^{\mathrm{I}}_i = \frac{1}{8\pi}\int_S f_j(\xi_1, \xi_2, \xi_3)(2R^{-1}\delta_{ij} - R_{,ij})\,\mathrm{d}\sigma \tag{18}$$

and

$${}_0q^{\mathrm{I}} = -\frac{1}{4\pi}\int_S f_j(\xi_1, \xi_2, \xi_3)R_{,j}\mathrm{d}\sigma. \tag{19}$$

In [18] and [19] δ_{ij} is the Kronecker delta and R is the distance from the point (ξ_1, ξ_2, ξ_3) on the surface S of the particle to a general point (x_1, x_2, x_3) in space, i.e.

$$R^2 = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2].$$

The integral in [18] and [19] is over the surface S of B. The appropriate distribution of forces (Stokeslets) $f_j(\xi_1, \xi_2, \xi_3)$ over S is obtained by solving the Fredholm-type surface integral equation,

$$\alpha_i + \alpha_{ij}x_j + \alpha_{ijk}x_kx_j + \cdots = \frac{1}{8\pi}\int_S f_j(\xi_1, \xi_2, \xi_3)(2\delta_{ij}R^{-1} - R_{,ij})\mathrm{d}\sigma, \tag{20}$$

for points (x_1, x_2, x_3) on S. It can be shown that f_j is related to the stress distribution $\bar{\sigma}_{jk}$ on S due to the zero-order inner field by (Eshelby 1959)

$$f_j = \bar{\sigma}_{jk}\mathbf{n}_k,$$

where $\mathbf{n}_k$ is the unit outward normal to S. Also,

$$U_{ij}\{x_1, x_2, x_3; \xi_1, \xi_2, \xi_3\} = \frac{1}{8\pi}(2R^{-1}\delta_{ij} - R_{,ij})$$

is the fundamental Stokeslet solution.

For large $|r|$, U_{ij} and R_j^{-1} have the following Taylor series expansions:

$$U_{ij} = s_{ij} - \xi_k s_{ij,k} + \tfrac{1}{2}\xi_l\xi_k s_{ij,kl} + \cdots \tag{21}$$

and

$$R_{,j}^{-1} = t_{,j} - \bar{\xi}_k t_{,jk} + \tfrac{1}{2}\bar{\xi}_l\bar{\xi}_k t_{,jkl} + \cdots, \qquad [22]$$

where

$$t^{-1} = |r| = [x_1^2 + x_2^2 + (x_3 - K^{-1})^2]^{1/2}, \qquad [23a]$$

$$s_{ij} = 2t\delta_{ij} - (t^{-1})_{,ij} \qquad [23b]$$

and

$$\bar{\xi}_k = \xi_k - \delta_{k3}K^{-1}. \qquad [23c]$$

In [21] and [22], we have made use of the fact that at $|r'| = 0$,

$$\frac{\partial^n}{\partial\xi_k\partial\xi_l\ldots\partial\xi_p}\{U_{ij};\, t_{,j}\}_{r_{,}=0} = (-1)^n(s_{ij,kl\ldots p};\, t_{,jkl\ldots p}). \qquad [24]$$

In [24], $|r'|$ is distance from O to a point (ξ_1, ξ_2, ξ_3), i.e. $|r'|^2 = \xi_i\xi_i$. Substituting [21] and [22] into [18] and [19], we have,

$$_0u_i = A_j s_{ij} + A_{jk}s_{ij,k} + A_{jkl}s_{ij,kl} + \cdots \qquad [25]$$

and

$$_0q_i = 2(A_j t_{,j} + A_{jk}t_{,jk} + A_{jkl}t_{,jkl} + \cdots), \qquad [26]$$

where

$$A_j = \frac{1}{8\pi}\int_S f_j \mathrm{d}\sigma,\; A_{jk} = -\frac{1}{8\pi}\int_S f_j\bar{\xi}_k\,\mathrm{d}\sigma$$

and

$$A_{jkl} = \frac{1}{16\pi}\int_S f_j\bar{\xi}_k\bar{\xi}_l\,\mathrm{d}\sigma. \qquad [27]$$

Note that A_j, A_{jk}, A_{jkl} etc. depend upon the shape, size and orientation of B. They also depend linearly on the velocity vector on B through [20]. In view of this it is possible to write A_j, A_{jk}, A_{jkl} etc. in the form

$$A_j = \alpha_i C_{ij} + \alpha_{ik}C_{kij} + \alpha_{ikl}C_{lkij} + \cdots, \qquad [28a]$$

$$A_{jk} = \alpha_i D_{ijk} + \alpha_{il}D_{lijk} + \alpha_{ilm}D_{mlijk} + \cdots, \qquad [28b]$$

$$A_{jkl} = \alpha_i G_{ijkl} + \alpha_{im}G_{mijkl} + \alpha_{imn}G_{nmijkl} + \cdots \text{ etc.} \qquad [28c]$$

When expressed in the outer variables [25] and [26] now become

$$_0u_i = KA_j\tilde{s}_{ij} + K^2A_{jk}\tilde{s}_{ij,k} + K^3A_{jkl}\tilde{s}_{ij,kl} + \cdots \qquad [29a]$$

and

$$\frac{_0q}{K^2} = 2(A_j\tilde{t}_{,j} + KA_{jk}\tilde{t}_{,jk} + K^2A_{jkl}\tilde{t}_{,jkl} + \cdots), \qquad [29b]$$

respectively; where $\tilde{s}_{ij}$ and $\tilde{t}_{,j}$ are defined by [23a–c] with $\tilde{r}$ and K replaced r and 1, respectively, while all the differential operations indicated in [29a, b] are to be done with respect to the outer independent variables.

3.2. *The first-order outer field*

The first-order outer field $({}_0u_i, {}_0\tilde{q})$ satisfies [14a–c]–[16]. In addition, it is constrained to have the same form as [29a, b] in its inner region ($r \to 0$) in order that the asymptotic matching conditions have a chance of being satisfied.

To construct this outer field, define the auxillary field $({}_1\tilde{u}_i^*, {}_1\tilde{q}^*)$ by

$${}_1\tilde{u}_i^1 = {}_1\tilde{u}_i^* + {}_0\tilde{u}_i^1 \quad \text{and} \quad \frac{{}_1\tilde{q}}{K} = \frac{{}_1q^*}{K} + \frac{{}_0\tilde{q}^1}{K}.$$

The general two-phase Stokeslet solutions due to Aderogba & Blake (1978) and Lee *et al.* (1979) are then used to determine the functional form of ${}_1\tilde{u}_i^*$, ${}_1\tilde{q}^*$, ${}_1\tilde{u}^{11}$ and ${}_1\tilde{q}^{11}$. Hence we have,

$${}_1\tilde{u}_i^* = {}_0\hat{u}_j J_{ij} + \tfrac{1}{2}(1 - \Gamma)(2\tilde{x}_{3\,0}\hat{u}_{3,i} + \tilde{x}_3^2 B_{ij\,0}\hat{u}_{j,kk}) \tag{30}$$

and

$${}_1\tilde{u}_i^{11} = (\Gamma + 1)(B_{ij\,0}\tilde{u}_j + x_{3\,0}\tilde{u}_{3,l} - \tfrac{1}{2}x_{3\,0}^2\tilde{u}_{i,kk}). \tag{31}$$

In [30] and [31],

$$\Gamma = (1-\lambda)(1+\lambda)^{-1}, J_{ij} = \Gamma\delta_{ij} - (\Gamma + 1)\delta_{i3}\delta_{j3},$$

$$B_{ij} = \delta_{ij} - 2\delta_{i3}\delta_{j3} \quad \text{and} \quad {}_0\tilde{u}_i(x_i, x_2, -x_3).$$

Preparatory to obtaining the solutions to the equations of the first-order inner fields, we expand ${}_1\tilde{u}_i^*$ in a Taylor series expansion about O for small values of r. The resulting expansion is

$${}_1\tilde{u}_i^* = K\{{}_1E_i + {}_1E_{im}\hat{x}_m + {}_1E_{imn}\hat{x}_n\hat{x}_m + \cdots\}, \tag{32}$$

where

$${}_1E_i = A_jP_{ji} + KA_{jk}P_{kji} + K^2A_{jkn}P_{nkji} + \cdots, \tag{33a}$$

$${}_1E_{im} = A_j\bar{P}_{jim} + KA_{jk}\bar{P}_{kjim} + \cdots, \tag{33b}$$

$${}_1E_{imn} = A_j\bar{\bar{P}}_{jimn} + KA_{jk}\bar{\bar{P}}_{kjimn} + \cdots \text{ etc.} \tag{33c}$$

and

$$\hat{x}_m = \bar{x}_m - \delta_{m3}.$$

The coefficients of A_j, A_{jk} etc. in [33a–c] are given by

$$P_{ji} = \beta_{jp}J_{pi} + \tfrac{1}{2}(1-\Gamma)(2\beta_{j3i} + \beta_{jpll}B_{pi}), \tag{34a}$$

$$P_{kji} = [\beta_{jpn}J_{pi} + \tfrac{1}{2}(1-\Gamma)(2\beta_{j3in} + \beta_{jplln}B_{pi})]B_{nk}, \tag{34b}$$

$$\bar{P}_{jim} = \beta_{jpm}J_{pi} + \tfrac{1}{2}(1-\Gamma)[2(\beta_{j3im} + \delta_{m3}\beta_{j3i}) + (\beta_{jpllm} + 2\delta_{m3}\beta_{jpll})B_{pi}], \tag{34c}$$

$$\bar{P}_{kjim} = \{\beta_{jpmn}J_{pi} + \tfrac{1}{2}(1-\Gamma)[2(\beta_{j3imn} + \delta_{m3}\beta_{j3in}) + (\beta_{jpllnm} + 2\delta_{m3}\beta_{jplln})B_{pi}]\}B_{nk} \text{ etc.} \tag{34d}$$

In [34a–d],

$$\beta_{jp} = \tilde{s}_{jp}(0, 0, -1) \quad \text{and} \quad \beta_{jpklm\ldots r} = s_{jp,klm\ldots r}(0, 0, -1).$$

Explicit expressions for some of the βs are

$$\beta_{jp} = \frac{\delta_{jp} + \delta_{j3}\delta_{p3}}{2}, \tag{35a}$$

$$\beta_{jpk} = \frac{\delta_{k3}\delta_{jp} + B_{kj}\delta_{p3} + B_{pk}\delta_{j3} + 3\delta_{j3}\delta_{p3}\delta_{k3}}{4} \tag{35b}$$

and

$$\beta_{jpkl} = \frac{\begin{array}{c}-\delta_{kl}\delta_{jp} + B_{jk}B_{pl} + B_{pk}B_{jl} + 3(\delta_{k3}\delta_{l3}\delta_{jp} + \delta_{l3}\delta_{p3}B_{jk} \\ + \delta_{l3}\delta_{j3}B_{pk} + \delta_{k3}\delta_{p3}B_{jl} + \delta_{j3}\delta_{k3}B_{pl} - \delta_{j3}\delta_{p3}\delta_{kl}) + 15\delta_{j3}\delta_{p3}\delta_{k3}\delta_{l3}\end{array}}{8}. \tag{35c}$$

In terms of the inner variables [32] takes the form

$${}_1\tilde{u}_i = K\,{}_1E_i + K^2\,{}_1E_{im}\bar{x}_m + K^3\,{}_1E_{imn}\bar{x}_n\bar{x}_m + \cdots, \tag{36}$$

where $\bar{x}_i = x_i - \delta_{i3}$.

3.3. First-order inner field

Apart from satisfying Stokes equations [7a–c], the first order inner field $({}_1u_i, {}_1q)$ should also satisfy the boundary conditions

$$ {}_1u_i = 0 \quad \text{on B} \tag{37a} $$

and

$$ {}_1u_i \to {}_1\tilde{u}_i = K\,{}_1E_i + K^2\,{}_1E_{im}\tilde{x}_m + K^3\,{}_1E_{lmn}\tilde{x}_n\tilde{x}_m + \cdots \quad \text{as } |r| \to \infty. \tag{37b} $$

The last condition [37b] ensures that the inner and outer first-order fields are properly matched to the first order in K.

Writing

$$ {}_1u_i = {}_1u_i^* + K({}_1E_i + K\,{}_1E_{im}\tilde{x}_m + K^2\,{}_1E_{imn}\tilde{x}_n\tilde{x}_m + \cdots), $$

it is easy to see that ${}_1u_i^*$ satisfies the boundary conditions

$$ {}_1u_i^* = -K({}_1E_i + K\,{}_1E_{im}\tilde{x}_m + K^2\,{}_1E_{imn}\tilde{x}_n\tilde{x}_m + \cdots) \quad \text{on B} $$

and

$$ {}_1u_i^* = 0 \text{ as } |r| \to \infty. $$

Following the procedure used in subsection 3.1 for constructing the zero-order inner field $({}_0u_i, {}_0q)$, it is straightforward to show that ${}_1u_i^*$ has the outer expansion (in the inner variables)

$$ {}_1u_i^* = K({}_1A_j s_{ij} + {}_1A_{jk}s_{jk,k} + {}_1A_{jkl}s_{ij,kl} + \cdots), \tag{38} $$

as $|r| \to \infty$.

Here,

$$ {}_1A_j = ({}_1E_iC_{ij} + K\,{}_1E_{im}C_{mij} + K^2\,{}_1E_{imn}C_{nmij} + \cdots), \tag{39a} $$

$$ {}_1A_{jk} = ({}_1E_iD_{ijk} + K\,{}_1E_{im}D_{mijk} + K^2\,{}_1E_{imn}\,D_{nmijk} + \cdots) \text{ etc.} \tag{39b} $$

Expressed in the inner variables, [38] has the form

$$ {}_1u_i^* = K^2({}_1A_j\tilde{s}_{ij} + K\,{}_1A_{jk}\tilde{s}_{ij,k} + K^2\,{}_1A_{jkl}\tilde{s}_{ij,kl} + \cdots). $$

3.4. Second-order outer field

We seek, as the second-order outer field, the pair $({}_2\tilde{u}_i, {}_2\tilde{q})$, which has the same behaviour as ${}_1u_i^*$ as $|r| \to 0$ and which satisfies [14a–c]–[16]. Following the procedure of subsection 3.2, we define the auxillary field, ${}_2u_i^*$, by

$$ {}_2\tilde{u}_i^{\text{I}} = {}_2\tilde{u}_i^* + {}_1\tilde{u}_i^*. $$

A repetition of the analyses given in subsection 3.2 immediately leads to

$$ {}_2\tilde{u}_i^* = J_{ij}\,{}_1\hat{u}_j^* + (1 - \Gamma)(\tilde{x}_3\,{}_1\hat{u}_{3,i}^* + \tfrac{1}{2}\tilde{x}_3^2\,{}_1\hat{u}_{j,kk}^*B_{ij}) \tag{40} $$

and

$$ {}_2\tilde{u}_i^{\text{II}} = (\Gamma + 1)(B_{ij}\,{}_1u_j^* + \tilde{x}_3\,{}_1u_{3,i}^* - \tfrac{1}{2}\tilde{x}_3^2\,{}_1u_{i,kk}^*). \tag{41} $$

In terms of the inner variables, ${}_2u_i^*$ has the Taylor series expansion

$$ {}_2\tilde{u}_i^* = K^2({}_2E_i + K\,{}_2E_{im}\hat{x}_m + K^2\,{}_2E_{imn}\hat{x}_n\hat{x}_m + \cdots), $$

as $|\tilde{r}| \to \infty$. The expressions for ${}_2E_i, {}_2E_{im}, {}_2E_{imn}$ etc. are obtained by using [39a, b] with A_j, A_{jk}, A_{jkl} etc. replaced by ${}_1A_j, {}_1A_{jk}, {}_1A_{jkl}$ etc.

3.5. Second-order inner field

Repeating the analysis of subsection 3.3 for this field, we obtain the following expansion for the second-order velocity ${}_2u_i$ as $|r| \to \infty$:

$${}_2u_i = K^2({}_2E_i + K_2E_{im}\bar{x}_m + K^2{}_2E_{imn}\bar{x}_n\bar{x}_m + \cdots) + K^2({}_2A_js_{ij} + {}_2A_{jk}s_{ij,k} + {}_2A_{jkl}s_{ij,kl} + \cdots),$$

where ${}_2A_j = ({}_2E_iC_{ij} + K_2E_{ik}C_{kij} + \cdots)$ and ${}_2A_{jk} = ({}_2E_iD_{ijk} + K_2E_{il}D_{lijk} + \cdots)$.

The procedure described in subsections 3.1–3.5 may be repeated *ad infinitum*.

3.6. *Force and torque on the particle*

The force and torque on the particle is obtained by applying the generalized Faxen's laws given by Brenner (1964b) to the inner fields. For the dimensionless force F_j and torque T_j (non-dimensionalized with respect to $\mu^1 U\mathbf{a}$ and $\mu^1 U\mathbf{a}^2$, respectively), we have the series (ordered in K)

$$\begin{aligned} F_j = 8\pi\,[&A_j - KA_kP_{ki}C_{ij} + K^2(A_kP_{ki}C_{im}P_{mn}C_{nj} - A_k\bar{P}_{kim}C_{mij} - A_{nk}P_{kni}C_{ij}) \\ &- K^3(A_kP_{kq}C_{qi}P_{im}C_{mo}P_{on}C_{nj} + A_k\bar{P}_{kimn}C_{nmi} + A_{nk}\bar{P}_{knim}C_{mij} + A_{nkl}P_{lkni}C_{ij} \\ &- A_kP_{kn}D_{nml}P_{lmi}C_{ij} - A_kP_{kn}C_{no}\bar{P}_{oim}C_{mij} \\ &- A_k\bar{P}_{kmn}C_{nmq}P_{qi}C_{ij}A_{nk}P_{knm}C_{mq}P_{qo}C_{oj}) + O(K^4)] \end{aligned} \quad [42]$$

and

$$\begin{aligned} T_j = 8\pi\epsilon_{jik}[&A_{ik} - KA_qP_{ql}D_{lik} + K^2(A_qP_{qn}C_{nm}P_{ml}D_{lik} - A_q\bar{P}_{qml}D_{lmik} - A_{qn}P_{nql}D_{lik}) \\ &- K^3(A_qP_{qr}C_{ro}P_{ol}C_{ln}P_{nm}D_{mik} + A_{qn}\bar{P}_{nqml}D_{lmik} + A_q\bar{P}_{qmln}D_{nlmik} + A_{nml}P_{lmnq}D_{qik} \\ &- A_qP_{qm}C_{mo}\bar{P}_{ojl}D_{ljik} - A_qP_{qm}D_{mno}P_{onj}D_{jik} - A_q\bar{P}_{qol}C_{lon}P_{nm}D_{mik} \\ &- A_{qn}P_{nql}C_{lo}P_{om}D_{mik}) + O(K^4)]. \end{aligned} \quad [43]$$

Note that $A_j, A_{jk}, A_{jkl}, \ldots, C_{ij}, C_{ijk}, \ldots$ are all vectors and tensors which are determinable from the solution of the unbounded flow integral equation [20] and [27] and [28a–c]. On the other hand, $P_{ij}, P_{jik}, \ldots, \bar{P}_{ijk}, \ldots$ and $\bar{\bar{P}}_{ijkl}, \ldots$ depend on the ratio of viscosities for the planar interface problems under consideration here. From the structure of [42] and [43], it is obvious that, to calculate the force and torque on the particle to order $K^n (n > 2)$, it is in general necessary to obtain a solution to [20] for the particle when it is immersed in an unbounded flow field whose velocity distribution is a polynomial of degree $n - 1$.

4. SOME SPECIAL CASES

In this section, we consider some interesting cases for which [42] and [43] reduce to more degenerate forms.

4.1. *Pure translation in a quiescent fluid*

If fluids I and II are at rest at infinity and if the particle translates without rotating in fluid I with velocity U_i, then in [11]

$$\alpha_i = U_i, \quad \alpha_{ij} = \alpha_{ijk} = \cdots = \alpha_{ijk\ldots r} = \mathbf{0}.$$

Under these circumstances,

$$A_j = U_iC_{ij}$$

where, as shown by Brenner (1962), C_{ij} is a symmetric tensor. Also,

$$A_{jk} = U_iD_{ijk}, \quad A_{jkl} = U_iD_{ijkl} \text{ etc.}$$

If the particle under consideration is orthotropic (e.g. an ellipsoid, a rectangular parallelepiped or any polyhedron), or if it possesses any form of axial symmetry, it may be deduced (Brenner 1964c) that provided O coincides with the centre of reaction of the particle

$$A_{jk} = \mathbf{0} \quad [44a]$$

Also,

$$C_{ijk} = \mathbf{0} = D_{ijk}, \tag{44b}$$

since the particle would experience neither a torque when translating in an infinite fluid nor a force when rotating in same. For orthotropic bodies, O then is the point of intersection of the three mutually perpendicular planes of symmetry while for non-orthotropic bodies of revolution O lies somewhere on the axis of symmetry.

It is obvious from [34a] and [35a–c] that P_{ij} is a diagonal matrix, i.e.

$$P_{ji} = \delta_{ij} Q_i \quad \text{(no sum)}. \tag{45}$$

For the class of bodies under discussion here, C_{ij} is also a diagonal matrix provided the body is oriented such that its planes or axes of symmetry coincide with coordinate planes or axes, respectively, i.e.

$$C_{ij} = \delta_{ij} Z_j \quad \text{(no sum on } j). \tag{46}$$

The consequence of [44a, b]–[46] is that for these special orientations, [42] and [43] reduce, respectively, to

$$F_j = \frac{8\pi A_j}{(1 + g_j K + d_j K^3) + O(K^4)} \tag{47a}$$

and

$$T_j = \frac{8\pi K^2 b_j}{(1 + e_j K) + O(K^4)} \quad \text{(no sum on } j), \tag{47b}$$

where

$$g_j = Q_j Z_j$$

and

$$d_j = \frac{(A_k \bar{P}_{kimn} C_{nmij} + A_{nkl} P_{lkni} C_{ij})}{A_j}, \tag{48a}$$

$$b_j = -\epsilon_{jik} A_n \bar{P}_{nml} D_{lmik} \tag{48b}$$

and

$$e_j = \frac{A_q P_{qm} C_{mj}}{A_j} \quad \text{(no sum on } j). \tag{48c}$$

The result in [47a, b] and [48a–c] may be shown to hold true for any interface shape which is symmetrical about an axis through O provided the orthotropic or axisymmetric body possesses fore- and aft-symmetry about this axis. However, for non-planar interfaces which satisfy this symmetry condition, expressions for P_{ij}, P_{ijk} etc. would be different from those given in [34a–d].

For a sphere, simple calculations show that

$$Z_j = \tfrac{3}{4} \quad \text{and} \quad A_j = \frac{3U_j}{4}, \tag{49a}$$

$$Q_j = \begin{cases} \dfrac{2 - 3\lambda}{4(1 + \lambda)} & \text{for } j \neq 3 \quad [49b] \\[2ex] \dfrac{-(2 + 3\lambda)}{2(1 + \lambda)} & \text{for } j = 3 \quad [49c] \end{cases}$$

and

$$\epsilon_{jik} D_{lmik} = \tfrac{1}{2} \epsilon_{jml}. \tag{49d}$$

The only non-zero elements of the tensor $\bar{P}_{nml}$ are (from [34c] and [35a–c])

$$\bar{P}_{232} = \bar{P}_{131} = \tfrac{1}{2} \bar{P}_{333} = -\bar{P}_{322} = -\bar{P}_{311} = \frac{2 + 3\lambda}{8(1 + \lambda)}$$

and

$$\bar{P}_{223} = \bar{P}_{113} = \frac{3\lambda - 2}{8(1+\lambda)}.$$

It follows immediately from [48a–c] and [49a–d] that

$$g_j = \begin{cases} \dfrac{3}{16}\dfrac{2-3\lambda}{1+\lambda} & \text{for } j \neq 3 \\[2ex] \dfrac{-3}{8}\dfrac{2+3\lambda}{1+\lambda} & \text{for } j = 3 \end{cases} \qquad [50]$$

and

$$b_j = -4\pi\epsilon_{jml}A_n\bar{P}_{nml} = \tfrac{3}{2}\pi(1+\lambda)^{-1}(U_1\delta_{j1} - U_2\delta_{j2}). \qquad [51]$$

Equations [50] and [51] are in complete agreement with the corresponding results of Lee *et al.* (1979). Calculation of d_j requires the determination of C_{nkij} which in turn requires that [20] be solved for the case where the disturbance velocity has the distribution

$$u_i = \alpha_{ijk}x_kx_j.$$

For a sphere, it can be shown that (Brenner 1966)

$$C_{ijnm} = \tfrac{1}{4}\delta_{ij}\delta_{nm}.$$

For sphere motion parallel to the interface along the x_1-axis, say,

$$d_j = (\bar{P}_{1133}C_{3311} + \bar{P}_{1122}C_{2211} + \bar{P}_{1111}C_{1111} + A_{1111}P_{1111} + A_{122}P_{2211} + A_{133}P_{3311})\delta_{j1},$$

where

$$P_{1111} = 2\bar{P}_{1111} = \frac{2+7\lambda}{16(1+\lambda)},$$

$$P_{2211} = 2\bar{P}_{1122} = \frac{5\lambda - 2}{16(1+\lambda)},$$

$$P_{3311} = 2\bar{P}_{1133} = \frac{1-\lambda}{4(1+\lambda)}$$

and

$$P_{3333} = 2\bar{P}_{3333} = -\tfrac{1}{2},$$

and A_{lmn} has the same numerical value as $\frac{1}{8}\delta_{mn}$. It therefore follows that

$$d_j = \frac{(2\lambda+1)(\delta_{j1}+\delta_{j2})}{16(1+\lambda)} \quad \text{and} \quad e_j = \frac{3(2-3\lambda)(\delta_{j1}+\delta_{j2})}{16(1+\lambda)}. \qquad [52a, b]$$

For motion normal to the interface,

$$d_j = (\bar{P}_{3333}C_{3333} + \bar{P}_{3322}C_{2233} + \bar{P}_{3311}C_{1133} + A_{333}P_{3333} + A_{322}P_{2233} + A_{311}P_{1133})\delta_{j3}$$
$$= \frac{-3}{[8(1+\lambda)]\delta_{j3}}. \qquad [52c]$$

It is not necessary to calculate e_j for this case because the numerator of T_j in [47b] is zero. The results in [52a–c] represent an extension, to third order in K, of the corresponding results given in Lee *et al.* (1979).

4.2. Pure rotation in a quiescent fluid

If the particle is rotating with angular velocity w_k in a two-phase fluid which is at rest at infinity, then

$$\alpha_{ij} = \epsilon_{ikj}w_k, \alpha_{ijk} = \cdots = \alpha_{ijk\ldots r} = 0. \qquad [53]$$

Also,

$$A_j = \epsilon_{imn} w_m C_{nij}, \quad A_{jk} = \epsilon_{imn} W_m D_{nijk} \text{ etc.}$$

For an orthotropic or axially symmetric body oriented such that its planes or axes of symmetry are parallel to coordinate planes or axes, respectively, we have in addition to [44b], [45] and [46], that if O and the centre of reaction coincide,

$$A_j = \mathbf{0} = A_{jkl}. \qquad [54]$$

As a consequence of these symmetry properties, [42] and [43] reduce to

$$F_j = -8\pi K^2 (A_{nq} P_{qni} C_{ij} + K A_{nq} P_{qnm} C_{mk} P_{k0} C_{0j}) + \mathrm{O}(K^4) \qquad [55]$$

and

$$T_j = 8\pi \epsilon_{jik}(A_{ik} - K^3 A_{nq} P_{qnml} D_{mljk}) + \mathrm{O}(K^4). \qquad [56]$$

$|w|a$ has been selected as the characteristic speed, where $|w|$ is understood to mean the magnitude of the angular velocity [i.e. $|w| = (w_i w_i)^{1/2}$] and a is the sphere radius. To prevent the body from translating, a force F_j must be exerted on it.

For a spherical particle it can be shown that

$$-8\pi A_{nq} P_{qni} C_{ij} = \frac{\frac{3}{2}\pi \epsilon_{jn3} w_n}{1+\lambda} \qquad [57a]$$

and

$$8\pi \epsilon_{jik}(A_{ik} - K^3 A_{nq} \bar{P}_{qnml} D_{mlik}) = 8\pi \left[\bar{w}_n - 1 + \Gamma \frac{\delta_{nj}}{8} + \frac{(\Gamma - 2) Y_{nj}}{16} \right] \qquad [57b]$$

In [57a, b],

$$Y_{nj} = \delta_{nj} - \delta_{n3}\delta_{j3} \quad \text{and} \quad \bar{w}_n = \frac{w_n}{|w|}.$$

These results are in accord with the corresponding ones given by Lee *et al.* (1979).

5. MOTION OF AN ARBITRARILY-ORIENTED ELLIPSOID

In this section, the utility of the general expressions ([42] and [43]) given in section 3 is demonstrated by calculating the force and torque on an arbitrarily-oriented ellipsoid translating and rotating in a two-phase fluid.

Consider an ellipsoidal particle with semi-axes of lengths $\bar{a}_1, \bar{a}_2$ and $\bar{a}_3$. Let the particle be momentarily positioned in fluid I such that its centre, O, is at $(0, 0, d)(d > 0)$ relative to a cartesian coordinate system (x_1, x_2, x_3) whose origin, Q, is on the planar interface (see Figure 2). The interface is the plane $x_3 = 0$ in this coordinate system. The orientations of the x_1- and x_2-axes are such that the x_1-axis is parallel to the $\bar{a}_1$-semi-axis of the ellipsoid while the x_2- and x_3-axes make an arbitrary angle θ with the ellipsoid's $\bar{a}_2$- and $\bar{a}_3$-semi-axes, respectively, in the counter-clockwise direction. Thus, if (e_1', e_2', e_3') and (e_1, e_2, e_3) are two right-handed triads of orthonormal vectors lying, one along the principal axes of the ellipsoid and the other along the (x_1, x_2, x_3) coordinate axes, respectively, then the nine direction cosines are given by

$$M_{jk} \equiv e_j \cdot e_k' \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Let us assume that the particle translates with an arbitrary velocity U_i and, at the same time, rotates about an arbitrary axis through O with angular speed w_i. By virtue of the linearity of the governing equations and boundary conditions, the resistance of the particle while performing this general motion may be determined by appropriately superposing the

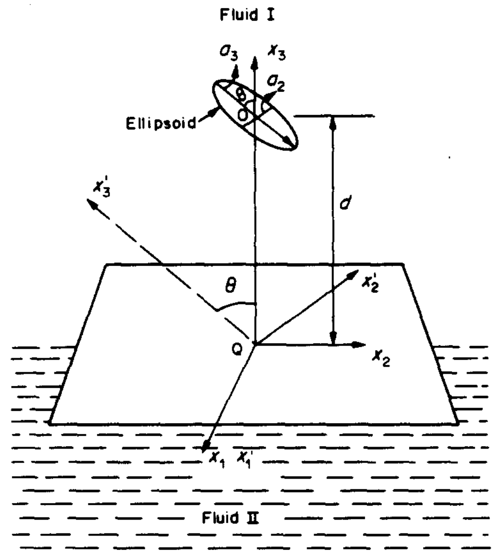

Figure 2. A schematic sketch of the positions of the arbitrarily-oriented ellipsoid and the interface in the cartesian coordinate systems (x_1, x_2, x_3) and (x_1', x_2', x_3'). The two coordinate systems are related by $x_k = M_{jk} x_j'$.

resistance of the particle for the six independent cases in each of which either the particle's direction of translational motion or its axis of rotation is parallel to one of the three coordinate axes. In the rest of this section, we determine, to at least order K^3, the force and torque on the particle in each of these six cases. Without loss of generality, we select $\bar{a}_3$ as the characteristic length of the problem with respect to which all other lengths are non-dimensionalized. Also, $K \equiv \bar{a}_3/d$.

5.1. Particle motion parallel to the interface along the x_1-axis

If the particle is moved parallel to the x_1-axis with dimensional speed U_1, then

$$\alpha_i = \delta_{i1},$$

where we have non-dimensionalized velocities with respect to U_1 as the characteristic speed.

From [42], it is evident that the calculation, to order K^3, of the particle drag requires the determination of only A_1, C_{11} and P_{11} since, by virtue of the orthotropicity of the particle, [44a, b] apply. The Stokeslet distribution f_j that satisfies [20] can be deduced from the results of Dyson (1891) and Eshelby (1959) to be

$$f_j = \epsilon_0 J_j \mathbf{n}_i \xi_i' \delta_{j1}, \qquad [58]$$

where

$$\epsilon_0 = 4(a_1 a_2)^{-1}, \qquad J_i = (I_0 + a_i^2 I_i)^{-1} \quad (\text{no sum on } i),$$

$$I_0 = \int_0^\infty \Delta(\psi)\,\mathrm{d}\psi, \quad I_i = \int_0^\infty (a_i^2 + \psi)^{-1} \Delta(\psi)\,\mathrm{d}\psi$$

and

$$\Delta(\psi) = [(a_1^2 + \psi)(a_2^2 + \psi)(1 + \psi)]^{-1/2}.$$

In [58], $\mathbf{n}_i$ is the outward unit normal to the ellipsoidal surface and (ξ_1', ξ_2', ξ_3') are the coordinates of a point on the particle surface in the system defined by the orthonormal triad (e_1', e_2', e_3') lying along the principal axes of the ellipsoid. Also, $a_1 = \bar{a}_1/\bar{a}_3$ and $a_2 = \bar{a}_2/\bar{a}_3$.

From [58], it follows that

$$A_j = 2J_1 \delta_{j1} \qquad [59a]$$

and, therefore,

$$C_{11} = 2J_1. \qquad [59b]$$

From [34a] and [35a–c], P_{ij} is determined to be the diagonal matrix given by

$$P_{ij} = \frac{\frac{1}{4}[(\delta_{ij} - \delta_{i3}\delta_{j3})(2 - 3\lambda) - 2\delta_{i3}\delta_{j3}(2 + 3\lambda)]}{1 + \lambda}. \qquad [60]$$

Applying [42], [43] and [59a], we obtain the components of the force, $F_j (j = 1, 2, 3)$, and torque, T_j, acting on the particle as

$$F_1 = 8\pi\mu^1\bar{a}_3 U_1(F_{110} + KF_{111} + K^2F_{112}) + O(K^3), \qquad [61]$$

$$F_2 = F_3 = 0,$$

$$T_1 = 0,$$

$$T_2 = -8\pi\mu^1\bar{a}_3^2 U_1 K^2 T_{12} + O(K^3) \qquad [62a]$$

and

$$T_3 = 8\pi\mu^1\bar{a}_3^2 U_1 K^2 T_{13} + O(K^3), \qquad [62b]$$

where

$$F_{110} = A_1, \quad F_{111} = -A_1 P_{11} C_{11}, \quad F_{112} = A_1(P_{11}C_{11}),$$

$$T_{12} = A_1\bar{P}_{113}(D_{3131} - D_{3113}) + A_1\bar{P}_{131}(D_{1331} - D_{1313})$$

and

$$T_{13} = A_1\bar{P}_{113}(D_{3112} - D_{3121}) + A_1\bar{P}_{131}(D_{1312} - D_{3121}).$$

Here,

$$D_{3131} = -\tfrac{2}{3}a_1^2(\mathrm{b}_1 \cos^2\theta + \mathrm{b}_2 \sin^2\theta), \qquad [63a]$$

$$D_{3113} = -\tfrac{2}{3}(a_3^2\mathrm{b}_3 \cos^2\theta + a_2^2\mathrm{b}_4 \sin^2\theta), \qquad [63b]$$

$$D_{1331} = -\tfrac{2}{3}a_1^2(\mathrm{b}_5 \cos^2\theta + \mathrm{b}_6 \sin^2\theta), \qquad [63c]$$

$$D_{1313} = -\tfrac{2}{3}(a_3^2\mathrm{b}_7 \cos^2\theta + a_2^2\mathrm{b}_8 \sin^2\theta), \qquad [63d]$$

$$D_{3121} = -\tfrac{2}{3}a_1^2(\mathrm{b}_2 - \mathrm{b}_1)\sin\theta\cos\theta, \qquad [63e]$$

$$D_{3112} = -\tfrac{2}{3}(a_2^2\mathrm{b}_4 - a_3^2\mathrm{b}_3)\sin\theta\cos\theta, \qquad [63f]$$

$$D_{1321} = -\tfrac{2}{3}a_1^2(\mathrm{b}_6 - \mathrm{b}_5)\sin\theta\cos\theta, \qquad [63g]$$

$$D_{1312} = -\tfrac{2}{3}(a_2^2\mathrm{b}_8 - a_3^2\mathrm{b}_7)\sin\theta\cos\theta, \qquad [63h]$$

$$\bar{P}_{113} = \bar{P}_{223} = \frac{3\lambda - 2}{8(1 + \lambda)} \qquad [64a]$$

and

$$\bar{P}_{131} = \bar{P}_{232} = \frac{2 + 3\lambda}{8(1 + \lambda)}. \qquad [64b]$$

In [63a–h], $\mathrm{b}_1, \mathrm{b}_2, \ldots, \mathrm{b}_8$ are constants whose values depend on the ratio $a_1:a_2:1$. The computed values of these constants for three different ratios $a_1:a_2:1$, namely 0.5:0.4:1 (case I), 0.5:0.6:1 (case II) and 0.5:0.8:1 (case III), are displayed in table 1.

From [61], it is seen that, to order K^3, F_1 is independent of the orientation angle, θ. However, higher-order terms introduce θ-dependence into F_1. In figures 3a and 3b, are plotted the variation of F_{111} and F_{112} with the viscosity ratio, λ. In figure 3a, it is seen that F_{111} is negative- or positive-valued according to whether λ is less or greater than $\frac{2}{3}$. As can be observed from figure 3b, F_{112}, like F_{110}, is positive-valued for all λ. A consequence of these observations is that, for $\lambda < \frac{2}{3}$, the force experienced by the particle in two-phase flow

Table 1. Values of the constants $b_1, b_2, \ldots, b_8$ for case I ($a_1:a_2:1 = 0.5:0.4:1$), case II ($a_1:a_2:1 = 0.5:0.6:1$) and case III ($a_1:a_2:1 = 0.5:0.8:1$)

Constants for the ellipsoids	Case I	Case II	Case III
b_1	0.6735	0.8240	0.9139
b_2	0.6877	1.0370	1.0360
b_3	0.0616	0.0819	0.0591
b_4	0.1457	0.2880	0.1517
b_5	0.2462	0.3210	0.3265
b_6	0.0931	0.4200	0.3886
b_7	0.4795	0.5484	0.6468
b_8	0.7402	0.9100	0.8003

is lower than that experienced in unbounded flow. Also, both F_{111} and F_{112} increase in magnitude as the particle size (volume) increases.

The qualitative dependence of T_2 on θ is shown in figure 4 for $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$. It is shown that T_2 increases in magnitude as particle size increases for any given pair of θ and λ. The rate of change of T_2 with respect to θ is also observed to be greatest for the smallest particle (case I) for a given value of λ. It is to be noted that, for $\lambda = 0$, T_2 is positive

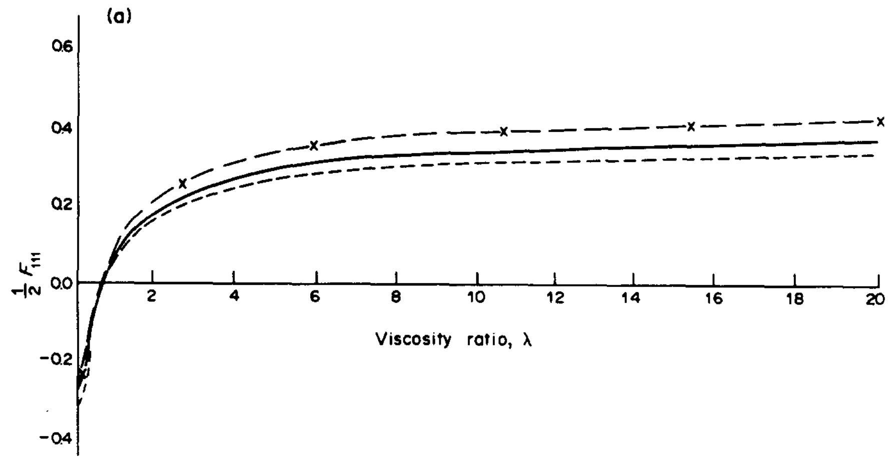

Figure 3a. First-order component, F_{111}, of the dimensionless drag force as a function of the viscosity ratio, λ, for an ellipsoid translating parallel to the x_1-axis: ---, case I ($a_1:a_2:1 = 0.5:0.4:1$); ——, case II ($a_1:a_2:1 = 0.5:0.6:1$); -- × --, case III ($a_1:a_2:1 = 0.5:0.8:1$).

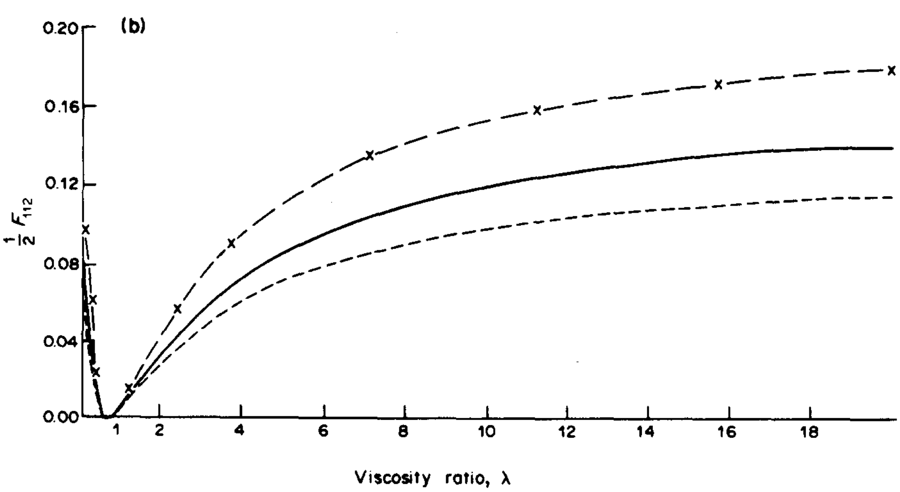

Figure 3b. Second-order component, F_{112}, of the dimensionless drag force as a function of the viscosity ratio, λ, for an ellipsoid translating parallel to the x_1-axis: ---, case I; ——, case II; -- × --, case III.

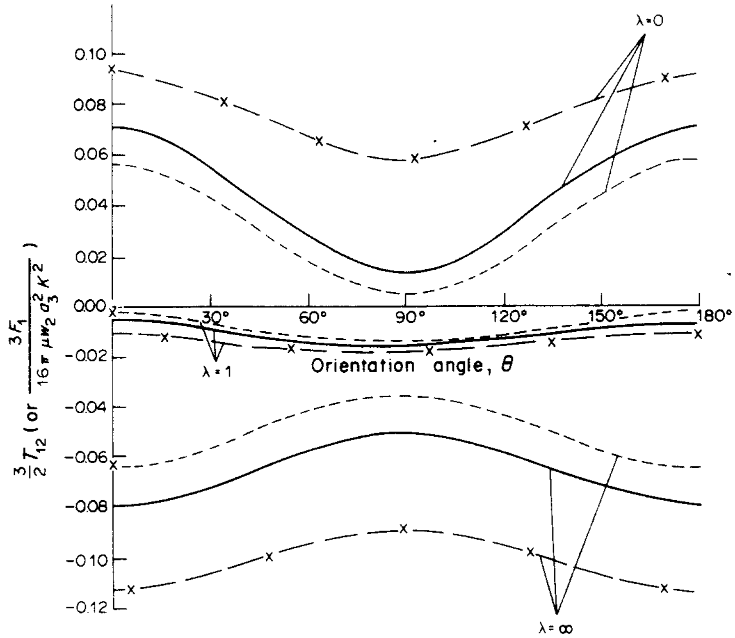

Figure 4. Dimensionless torque, $\frac{3}{2}T_{12}$ [or force $3F_1/(16\pi\mu^1 w_2\bar{a}_3^2K^2)$], as a function of orientation angle, θ for an ellipsoid translating parallel to the x_1-axis (or rotating with angular velocity w_2e_2): ---, case I; ——, case II; -- × --, case III.

while, for $\lambda = 1$ and $\lambda = \infty$, the torque is negative. Figure 5 shows the variation of torque, T_3, with θ for $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$ for all three cases. The direction of the torque for $\lambda = 0$ is again opposite to that for $\lambda = 1$ and $\lambda = \infty$. It is also worth noting that the magnitude of T_3 is about one order of magnitude less than that of T_2 for a given pair of λ and θ, and that T_3 decreases as the particle size (volume) increases.

5.2. Particle motion parallel to the interface along the x_2-axis

Next we calculate, to order K^3, the resistance of the ellipsoidal particle when it is moved parallel to the x_2-axis with dimensional speed U_2. We select U_2 as the characteristic speed

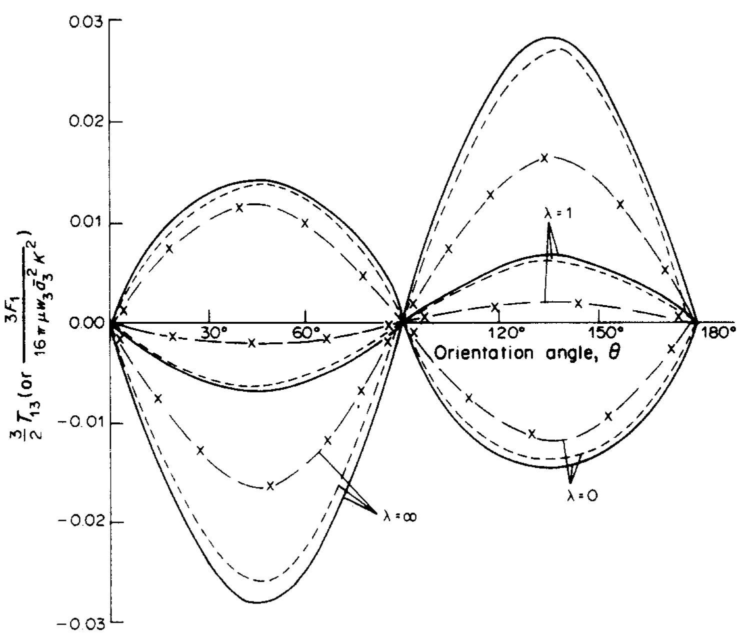

Figure 5. Dimensionless torque, $\frac{3}{2}T_{13}$ [or force $3F_1/(16\pi\mu^1 w_3\bar{a}_3^2K^2)$], as a function of orientation angle, θ for an ellipsoid translating parallel to the x_1-axis (or rotating with angular velocity w_3e_3): ---, case I; ——, case II, -- × --, case III.

of the flow field. We then have

$$\alpha_i = \delta_{i2}.$$

In this case, the Stokeslet distribution f_j is given by

$$f_j = \epsilon_0\{[(J_2\cos^2\theta + J_3\sin^2\theta)\delta_{j2} + (J_2 - J_3)\sin\theta\cos\theta_{j3}]\}\,\mathbf{n}_i\xi_i'. \qquad [65]$$

From [65], it follows that

$$A_j = 2[(J_2\cos^2\theta + J_3\sin^2\theta)\delta_{j2} + (J_2 - J_3)\sin\theta\cos\theta\delta_{j3}],$$

$$C_{22} = 2(J_2\cos^2\theta + J_3\sin^2\theta)$$

and

$$C_{23} = 2(J_2 - J_3)\cos\theta\sin\theta.$$

Since, as previously pointed out, C_{ij} is symmetric, we have

$$C_{23} = C_{32} \quad\text{and}\quad C_{21} = C_{12} = C_{31} = C_{13} = 0.$$

It may also be deduced from the unbounded flow solution in the case when the particle translates parallel to the x_3-axis that,

$$C_{33} = 2(J_3\cos^2\theta + J_2\sin^2\theta).$$

From [42] and [43], we obtain the following expressions for the force, F_j, and the torque, T_j, acting on the particle:

$$F_1 = 0, \qquad [66a]$$

$$F_2 = 8\pi\mu^1\bar{a}_3U_2(F_{220} + KF_{221} + K^2F_{222}) + O(K^3), \qquad [66b]$$

$$F_3 = 8\pi\mu^1\bar{a}_3U_2(F_{230} + KF_{231} + K^2F_{232}) + O(K^3), \qquad [66c]$$

$$\begin{aligned} T_1 = &-8\pi\mu^1\bar{a}_3^2U_2\{A_2[\bar{P}_{232}(D_{2323} - D_{2332}) + \bar{P}_{223}(D_{3223} - D_{3232})] \\ &+ A_3[\bar{P}_{311}(D_{1123} - D_{1132}) + \bar{P}_{322}(D_{2223} - D_{2232}) \\ &+ \bar{P}_{333}(D_{3323} - D_{3332})]\}K^2 + O(K^3) \end{aligned} \qquad [66d]$$

and

$$T_2 = T_3 = 0. \qquad [66e]$$

In [66a–e],

$$F_{220} = A_2, \quad F_{221} = -(A_2P_{22}C_{22} + A_3P_{33}C_{32}) \qquad [67a, b]$$

$$F_{222} = A_2P_{22}(C_{22}P_{22}C_{22} + C_{23}P_{33}C_{32}) + A_3P_{33}(C_{32}P_{22}C_{22} + C_{33}P_{33}C_{32}), \qquad [67c]$$

$$F_{230} = A_3, \quad F_{231} = -(A_2P_{22}C_{23} + A_3P_{33}C_{33}), \qquad [67d, e]$$

$$F_{232} = A_2P_{22}(C_{22}P_{22}C_{23} + C_{23}P_{33}C_{33}) + A_3P_{33}(C_{32}P_{22}C_{23} + C_{33}P_{33}C_{33}), \qquad [67f]$$

$$\bar{P}_{333} = -2\bar{P}_{322} = -2\bar{P}_{311} = \frac{-(2+3\lambda)}{4}, \qquad [68]$$

$$D_{3223} - D_{3232} = \tfrac{2}{3}[a_3^2(d_1\sin^2\theta + d_2\cos^2\theta) + a_2^2(d_3\sin^2\theta + d_4\cos^2\theta)], \qquad [69a]$$

$$D_{2323} - D_{2332} = -\tfrac{2}{3}[a_3^2(d_1\cos^2\theta + d_2\sin^2\theta) + a_2^2(d_3\cos^2\theta + d_4\sin^2\theta)], \qquad [69b]$$

and

$$(D_{1123} - D_{1132}) + (D_{2223} - D_{2232}) - 2(D_{3323} - D_{3332}) = \tfrac{2}{3}(a_3^2d_5 + a_2^2d_6)\cos\theta\sin\theta. \qquad [69c]$$

The constants $d_1, d_2, \ldots, d_6$ appearing in [69a–c] depend on the ratio $a_1 : a_2 : 1$. The computed values of these constants for cases I, II and III are displayed in table 2. The grouping of terms on the l.h.s. of [69a–c] is guided by the combinations in which these terms appear in [66d].

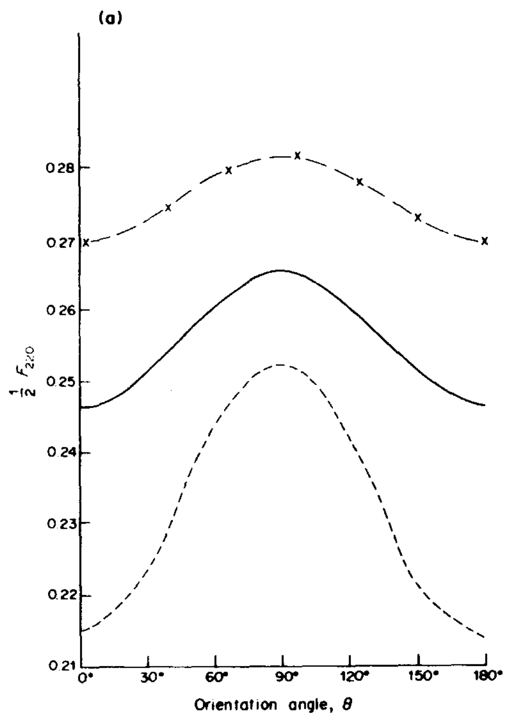

Figure 6a. Zero-order component, F_{220}, of the dimensionless drag force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_2-axis: ---, case I; ——, case II; -- × --, case III.

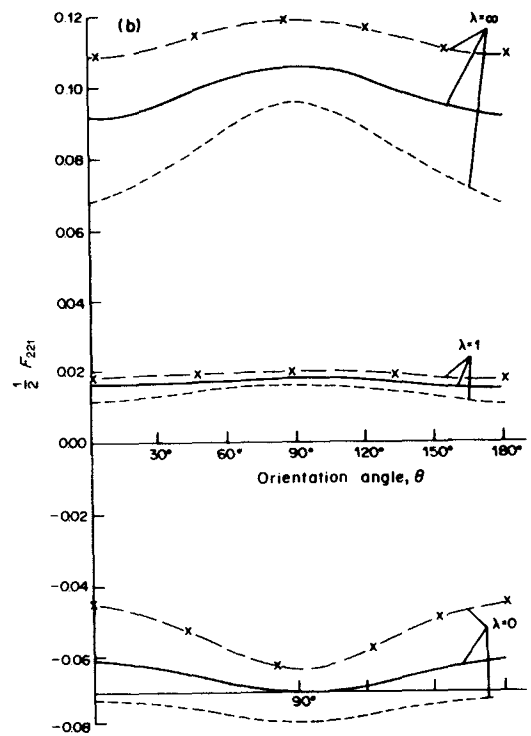

Figure 6b. First-order component, F_{221}, of the dimensionless drag force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_2-axis: ---, case I; ——, case II; -- × --, Case III.

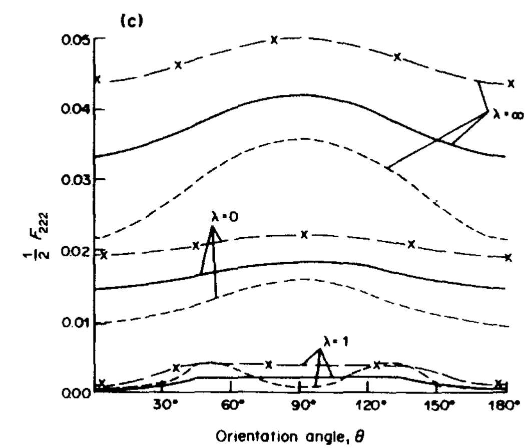

Figure 6c. Second-order component, F_{222}, of the dimensionless drag force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_2-axis: ---, case I; ——, case II; -- × --, Case III.

Table 2. Values of the constants $d_1, d_2, \ldots, d_6$ for cases I, II and III

Constants for the ellipsoids	Case I	Case II	Case III
d_1	0.0450	0.0720	0.1112
d_2	−0.4832	−0.5404	−0.6088
d_3	−0.7197	−0.6685	−0.6175
d_4	0.2814	0.2044	0.1738
d_5	−1.5846	−1.8372	−2.160
d_6	3.0033	2.6187	2.5359

In figures 6a–6c are shown the qualitative variations with θ of F_{220}, F_{221} and F_{222}, respectively, for $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$. It is observed in figures 6a–6c that the zero-, first- and second-order components of F_2 all increase in magnitude as particle size (volume) increases (with the exception of F_{222} for case III, $\lambda = 1$) for a given pair of θ and λ. The rate of variation of these ordered components with the orientation angle θ is found to be greatest for the smallest particle. For $\lambda = 0$, F_{221} is negative-valued for all orientations and all three particles. With the exception of this latter case, all the other quantities plotted in figures 6a–6c are positive-valued. From this observation, it is concluded that, regardless of particle size and orientation, the effect of a free surface is to decrease the drag on a particle relative to the corresponding unbounded fluid drag when the direction of particle motion is parallel to the interface. For $\lambda = 1$ and $\lambda = \infty$, the drag is increased over its unbounded fluid value. It should also be noted that, with the exception of F_{222} for $\lambda = 1$, all the ordered components of F_2 increase monotonically in value in the range $0° < \theta < 90°$ and decrease monotonically in the range $90° \leqslant \theta \leqslant 180°$.

Figures 7a–7c show the variation of F_{230}, F_{231} and F_{232} with θ for $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$. It is seen from these figures that in contrast to those of F_2, the ordered components of F_3 increase in magnitude as particle size decreases for any given pair of λ and θ, except at $\theta = 0°$, $\theta = 90°$ and $\theta = 180°$ where all components are zero-valued. All components have same sign at any given value of θ and have their largest magnitudes at a value of θ which is slightly less than 45° or slightly greater than 135°.

As can be seen in figure 8, where $3T_1/16\pi\mu U_2\bar{a}_3^2$ is plotted against θ, T_1 is negative-valued

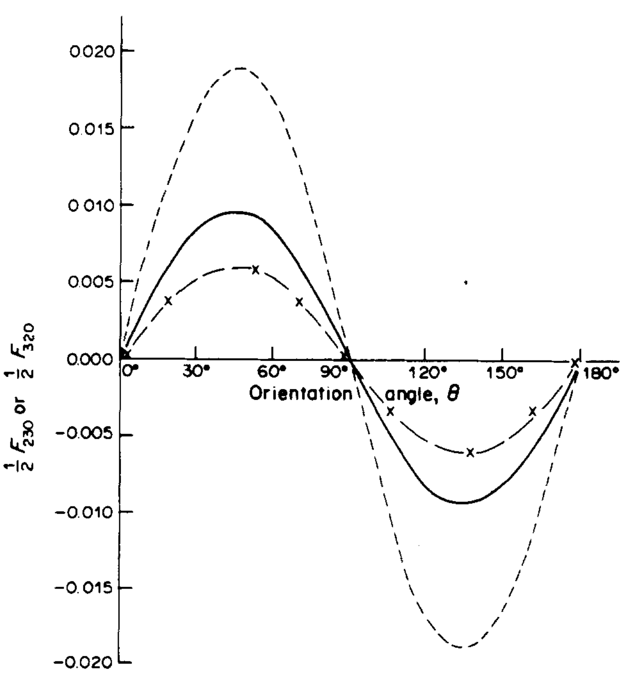

Figure 7a. Zero-order component, F_{230}, (or F_{320}) of the normal force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_2-axis (or x_3-axis): ---, case I; ——, case II; -- x --, case III.

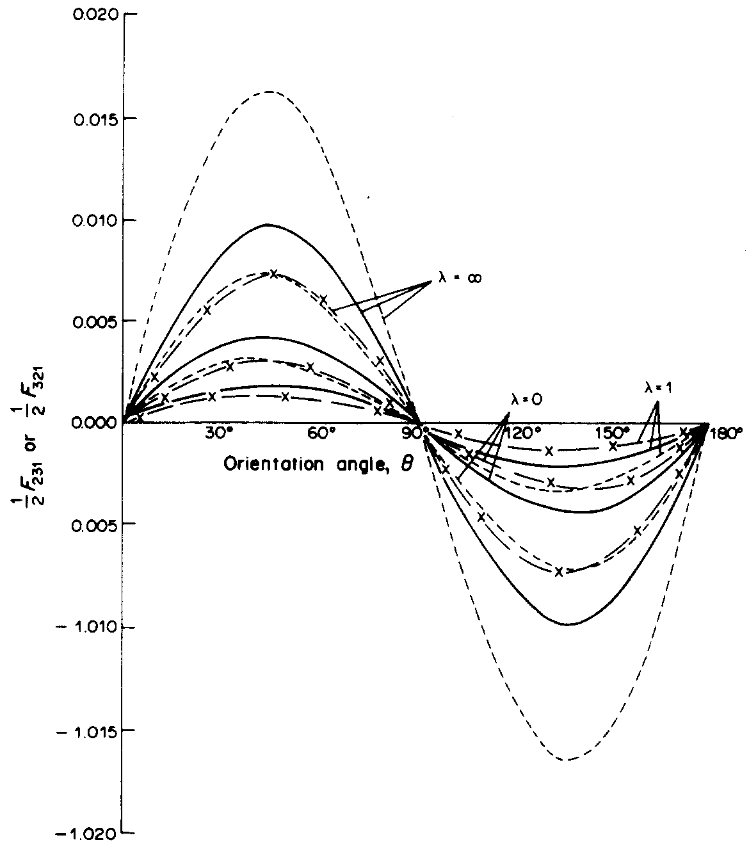

Figure 7b. First-order component, F_{231}, (or F_{321}) of the normal force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_2-axis (or x_3-axis): ---, case I; ——, case II; -- × --, case III.

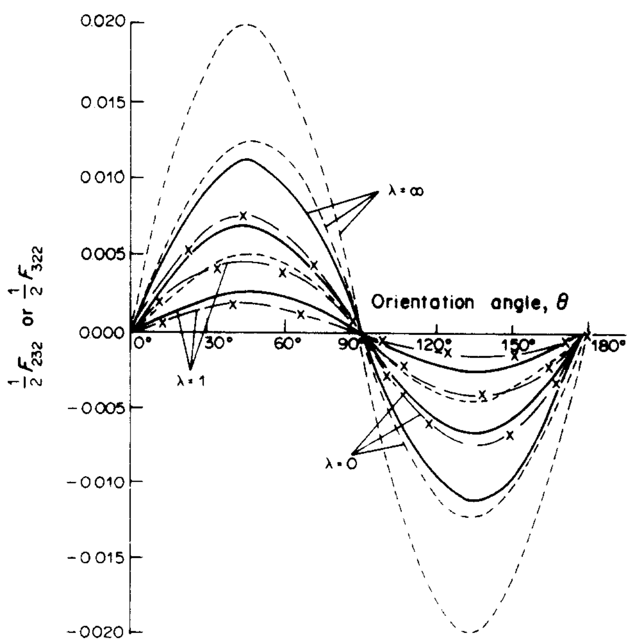

Figure 7c. Second-order component, F_{232}, of the normal force as a function of the orientation angle, θ for for an ellipsoid translating parallel to the x_2-axis (or x_3-axis): ---, case I; ——, case II; -- × --, case III.

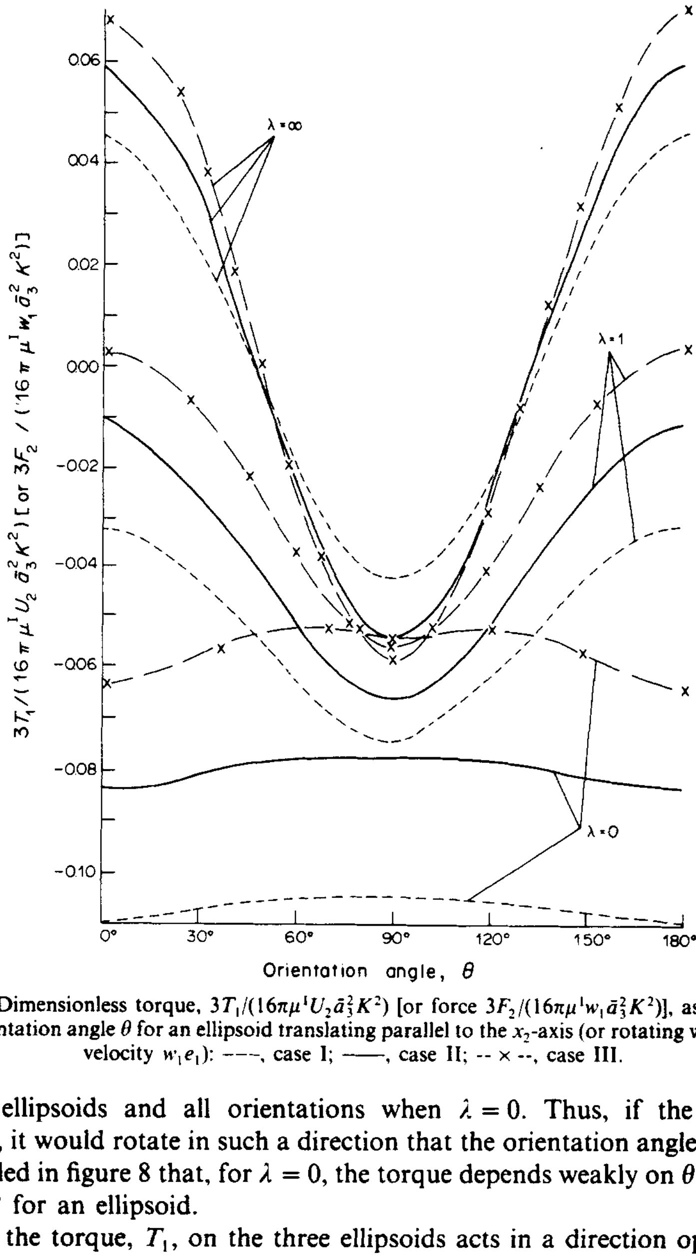

Figure 8. Dimensionless torque, $3T_1/(16\pi\mu^1 U_2\bar{a}_3^2K^2)$ [or force $3F_2/(16\pi\mu^1 w_1\bar{a}_3^2K^2)$], as a function of the orientation angle θ for an ellipsoid translating parallel to the x_2-axis (or rotating with angular velocity w_1e_1): ----, case I; ——, case II; -- × --, case III.

for all three ellipsoids and all orientations when $\lambda = 0$. Thus, if the particle were unconstrained, it would rotate in such a direction that the orientation angle θ is increased. It is also revealed in figure 8 that, for $\lambda = 0$, the torque depends weakly on θ over the range $60° < \theta < 120°$ for an ellipsoid.

For $\lambda = \infty$, the torque, T_1, on the three ellipsoids acts in a direction opposite to that for $\lambda = 0$ when $\theta < \theta_1$ or $\theta > 180 - \theta_1$, where θ_1 depends on the ratio $a_1 : a_2 : 1$. For the three ellipsoids the largest positive torque occurs for $\theta = 0°$, while the largest negative torque occurs for $\theta = 90°$. Again, the rate of change of T_1 with respect to θ is greatest for the smallest particle. Figure 8 also suggests that for a sufficiently "slender" ellipsoid ($a_3 > a_2, a_3 > a_1$) a change in the sign of the torque may be obtained as the orientation angle θ changes from 0° to 90° (or from 90° to 180°), provided the order of magnitude of λ is $\geqslant 1$.

5.3. Motion normal to the interface

In this case, the ellipsoid is presumed to be moving perpendicular to the interface with the speed U_3. Choosing U_3 as the characteristic speed, we have

$$\alpha_i = \delta_{i3}.$$

For this case, we also have

$$f_j = \epsilon_0[(J_3\cos^2\theta + J_2\sin^2\theta)\delta_{j3} + (J_2 - J_3)\sin\theta\cos\theta\,\delta_{j2}]\mathbf{n}_i\xi_i' \qquad [70]$$

and

$$A_j = 2[(J_2 - J_3)\cos\theta\sin\theta\,\delta_{j2} + (J_3\cos^2\theta + J_2\sin^2\theta)\delta_{j3}]. \qquad [71]$$

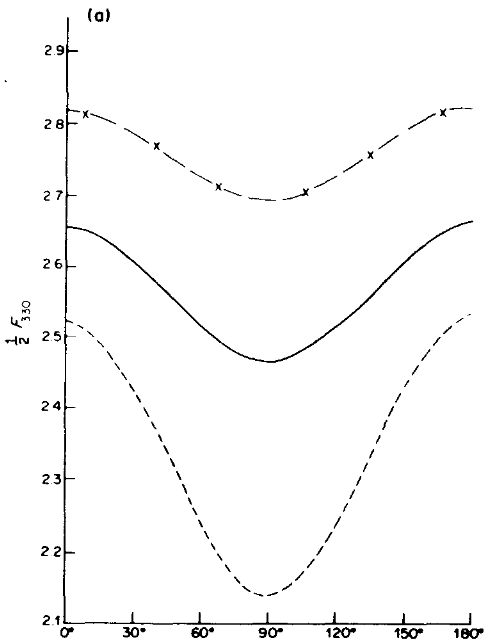

Figure 9a. Zero-order component, F_{330}, of the drag force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_3-axis: ---, case I; ——, case II; -- × --, case III.

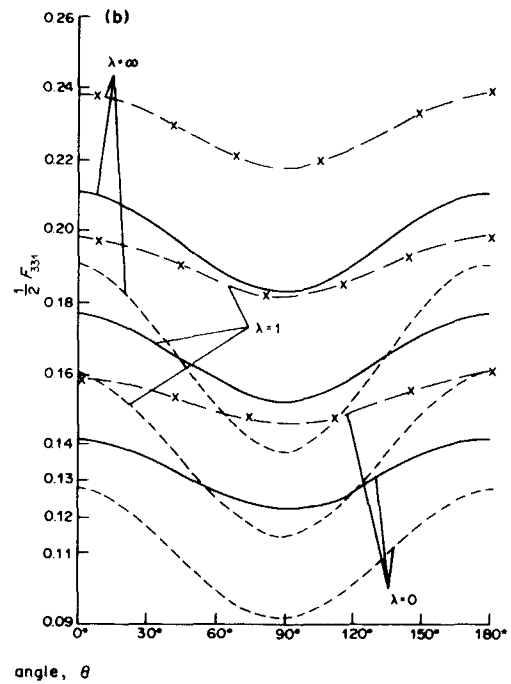

Figure 9b. First-order component, F_{331}, of the drag force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_3-axis: ---, case I; ——, case II; -- × --, case III.

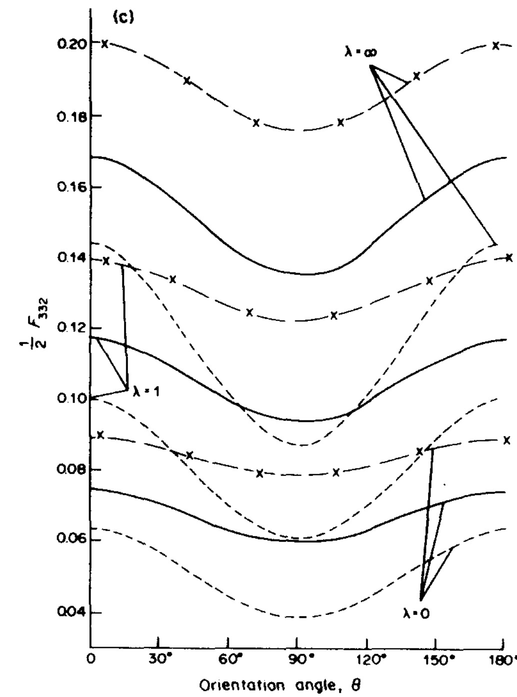

Figure 9c. Second-order component, F_{332}, of the drag force as a function of the orientation angle θ for an ellipsoid translating parallel to the x_3-axis: ---, case I; ——, case II; -- × --, case III.

The force, F_j, and torque, T_j, acting on the particle may be expressed as

$$F_1 = 0, \quad F_2 = 8\pi\mu^1\bar{a}_3U_3(F_{320} + KF_{321} + K^2F_{322}) + O(K^3), \qquad [72a, b]$$

$$F_3 = 8\pi\mu^1\bar{a}_3U_3(F_{330} + KF_{331} + K^2F_{332}) + O(K^3) \qquad [72c]$$

and

$$T_1 = -8\pi\mu^1\bar{a}_3U_3\{A_2[\bar{P}_{232}(D_{2323} - D_{2332}) + \bar{P}_{223}(D_{3223} - D_{3232})] + A_3[\bar{P}_{311}(D_{1123} - D_{1123}) + \bar{P}_{322}(D_{2233} - D_{2232}) + \bar{P}_{333}(D_{3323} - D_{3332})]\}K^2 + O(K^3). \qquad [72d]$$

Expressions for F_{320}, F_{321}, F_{322}, F_{330}, F_{331} and F_{332} are, respectively, given by the r.h.s. of [67a–f]. Note, however, that the operative definitions of A_2 and A_3 in this section are those given by [71]. The expressions for F_{320}, F_{321} and F_{322} turn out to be identical to those for F_{230}, F_{231} and F_{232}, respectively, to order K^3. The qualitative variations of F_{330}, F_{331} and F_{332} with θ for the three ellipsoids and for $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$ are plotted in figures 9a–9c. From these figures we deduce that the three ordered components of F_3 are all positive and they increase as particle size increases for any given pair of λ and θ. For any given λ, the rate of change of each of the three ordered components with θ decreases with particle size.

As can be seen from figure 10, the magnitude of T_1 increases with particle size for a given pair of λ and θ, except at $\theta = 0°$ and $\theta = 180°$ when $T_1 = 0$. Figure 10 also suggests that T_1 also increases with λ for a given particle size and orientation angle θ.

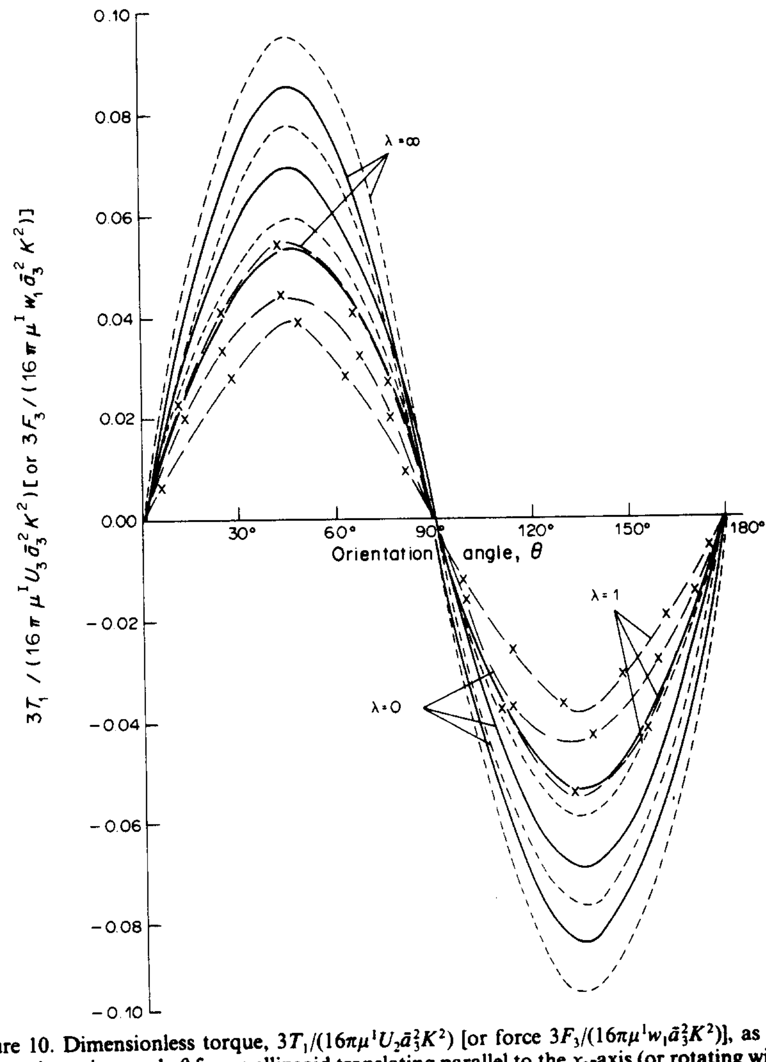

Figure 10. Dimensionless torque, $3T_1/(16\pi\mu^1U_2\bar{a}_3^2K^2)$ [or force $3F_3/(16\pi\mu^1w_1\bar{a}_3^2K^2)$], as a function of the orientation angle θ for an ellipsoid translating parallel to the x_3-axis (or rotating with angular velocity w_1e_1): ---, case I; ——, case II; -- × --, case III.

5.4. Rotation about an axis parallel to the x_1-axis

Next we consider an ellipsoid rotating with angular speed w_1 about the axis through O, which is parallel to the x_1-axis. Selecting $w_1 a_3$ as the characteristic speed of the flow field, we have

$$\alpha_{32} = 1 \quad \text{and} \quad \alpha_{23} = -1.$$

Also,

$$A_j = 0 \quad (j = 1, 2, 3),$$

while the only non-zero components of A_{jk} are given by

$$A_{23} = -\tfrac{2}{3}(\mathrm{d}_1 + \mathrm{d}_2) \quad \text{and} \quad A_{32} = \tfrac{2}{3}(\mathrm{d}_3 + \mathrm{d}_4),$$

where $\mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_8$ are given in table 2.

From [43], we have for the torque T_j on the particle,

$$T_1 = -8\pi\mu^1 \bar{a}_3^3 (T_{110} + K^3 T_{111}) + \mathrm{O}(K^4) \qquad [73]$$

and $T_2 = T_3 = 0$, where

$$T_{110} = A_{32} - A_{23}, \qquad [74a]$$

$$T_{111} = (A_{32}\bar{P}_{2323} + A_{23}\bar{P}_{3223})(D_{3223} - D_{3232}) + (A_{32}\bar{P}_{2332} + A_{23}\bar{P}_{3232})(D_{2323} - D_{2332}), \qquad [74b]$$

$$\bar{P}_{2323} = \bar{P}_{1313} = \bar{P}_{3131} = \bar{P}_{2323} = \tfrac{1}{4},$$

$$\bar{P}_{3223} = P_{3113} = -\frac{1 - 2\lambda}{4(1 + \lambda)} \quad \text{and} \quad \bar{P}_{2332} = \bar{P}_{1331} = \frac{2 + 5\lambda}{4(1 + \lambda)}. \qquad [74c]$$

The non-zero components, F_2 and F_3, of the force experienced by the particle are obtained to order K^3 by, respectively, multiplying the r.h.s. of [66e] by w_1/U_2 and the r.h.s. of [72d] by w_1/U_3. It is obvious from [74a] that the zero-order torque, T_{110}, does not depend on θ. The third-order component of T_1, however, depends on θ in the manner shown qualitatively in figure 11. For all three ellipsoids and all values of λ and θ plotted, we observe that T_{111} is positive, implying that the presence of an interface causes an increase in the torque relative to its value for the particle in an infinite fluid. It is also to be noted that for given values of λ and θ, the magnitude of T_{111} increases with particle size.

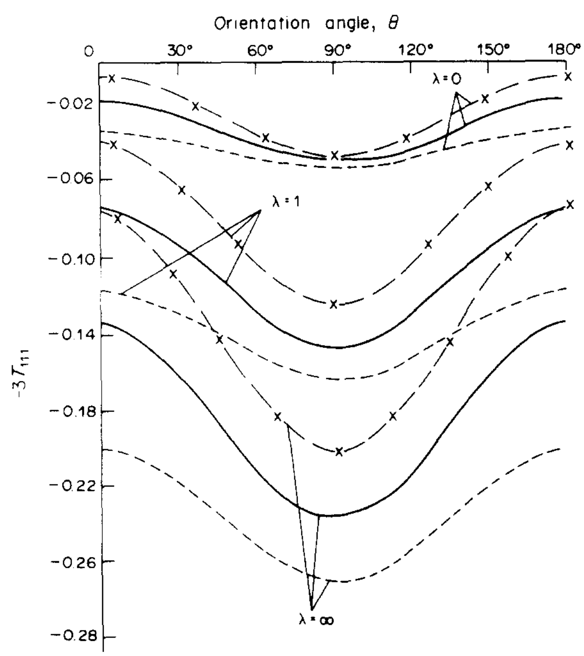

Figure 11. Dimensionless third-order torque, T_{111}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity $w_1 e_1$: ---, case I; ——, case II; -- × --, case III.

5.5. Rotation about an axis parallel to the x_2-axis

Here, $w_2\bar{a}_3$ is selected as the characteristic speed of flow, where w_2 is the angular speed of particle rotation about the axis through O which is parallel to the x_2-axis. The only non-zero elements of α_{ij} in this situation are

$$\alpha_{13} = 1 \quad \text{and} \quad \alpha_{31} = -1.$$

We deduce from [43] that

$$T_2 = -8\pi\mu^1 w_2 \bar{a}_3^3 (T_{220} + K^3 T_{221}) + O(K^4) \qquad [75a]$$

and

$$T_3 = -8\pi\mu^1 w_2 \bar{a}_3^3 (T_{230} + K^3 T_{231}) + O(K^4). \qquad [75b]$$

Here,

$$T_{220} = A_{13} - A_{31}, \quad T_{230} = A_{21} - A_{12}, \qquad [76a, b]$$

$$T_{221} = -(A_{13}\bar{P}_{3131} + A_{31}\bar{P}_{1331})(D_{1313} - D_{1331}) - (A_{13}\bar{P}_{3113} + A_{31}\bar{P}_{1313})(D_{3113} - D_{3131}), \qquad [76c]$$

$$T_{231} = (A_{21}\bar{P}_{1212} + A_{12}\bar{P}_{2112})(D_{2112} - D_{2121}) + (A_{21}\bar{P}_{1221} + A_{12}\bar{P}_{2121})(D_{1212} - D_{1221}) \qquad [76d]$$

$$A_{13} = D_{3113} - D_{1313}, A_{31} = D_{3131} - D_{1331}, \qquad [77a, b]$$

$$A_{21} = D_{3121} - D_{1321}, A_{12} = D_{3112} - D_{1312}, \qquad [77c, d]$$

$$D_{1212} = -\tfrac{4}{3}(b_4 a_2 \cos^2\theta + b_3 a_3^2 \sin^2\theta), \qquad [78a]$$

$$D_{2112} = -\tfrac{4}{3}(b_8 a_2^2 \cos^2\theta + b_7 a_3^2 \sin^2\theta), \qquad [78b]$$

$$D_{1221} = -\tfrac{4}{3}a_1^2(b_2 \cos^2\theta + b_1 \sin^2\theta), \qquad [78c]$$

$$D_{2121} = -\tfrac{4}{3}a_1^2(b \cos^2\theta + b_5 \sin^2\theta), \qquad [78d]$$

$$\bar{P}_{1212} = \bar{P}_{2121} = \frac{2+\lambda}{16(1+\lambda)} \quad \text{and} \quad \bar{P}_{1221} = \bar{P}_{2112} = \frac{5\lambda - 2}{16(1+\lambda)}. \qquad [79a, b]$$

All other terms appearing in [75a, b] have been previously defined.

The expression for the only non-zero component, F_1, of force is obtained, to order K^3, by multiplying the r.h.s. [62a] by w_2/U_1.

The qualitative variations of T_{220} and T_{221} with θ are displayed in figures 12a and 12b, while those of T_{230} and T_{231} are displayed in figures 13a and 13b. It is seen from figures 12a and 13a that T_{220} increases and T_{230} decreases as the particle size increases for any given value θ. Figure 12b shows that T_{221} is positive for all values of θ and λ plotted but figure 13b shows that the sign of T_{231} for $\lambda = 0$ is opposite that for $\lambda = 1$ or $\lambda = \infty$ at any orientation angle θ.

5.6. Rotation about an axis normal to the interface

Consider next the rotation of the ellipsoidal particle about the axis through O which is normal to the interface. The characteristic velocity is chosen to be $w_3\bar{a}_3$ in this case. Thus, the only non-zero components of α_{ij} are

$$\alpha_{21} = 1 \quad \text{and} \quad \alpha_{12} = -1.$$

The expressions for T_2 and T_3 are as given in [75a, b], [76a–d], [78a–d] and [79a, b] (with w_2 replaced by w_3) except that now A_{12}, A_{21}, A_{31} and A_{13} are given by

$$A_{21} = D_{1221} - D_{2121}, \quad A_{12} = D_{1212} - D_{2112}, \qquad [80a, b]$$

$$A_{31} = D_{1231} - D_{2131} \quad \text{and} \quad A_{13} = D_{1213} - D_{2113}. \qquad [80c, d]$$

The zero-order component of T_2 is equal to T_{230} of [76a]. The latter has been plotted in figure 13a. The zero- and third-order components of T_3 are plotted as functions of θ in figures 14a and 14b. It is shown in these figures that the magnitude of the ordered components of T_3 for any given pair, λ and θ, increases as does the particle size. However, the sign of T_{331}, the third-order component of T_3 for $\lambda = 0$ is opposite that for $\lambda = 1$ or $\lambda = \infty$ for a given value of θ. Moreover, as shown in figure 15, T_{321}, the third-order

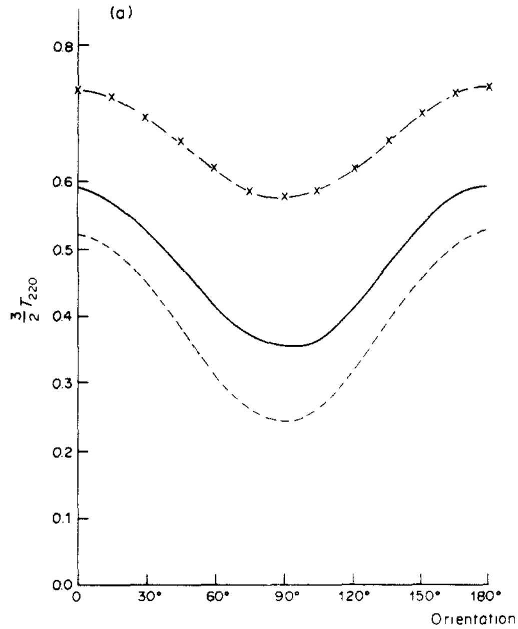

Figure 12a. Dimensionless zero-order torque, T_{220}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity $w_2 e_2$: ---, case I; ——, case II; -- × --, case III.

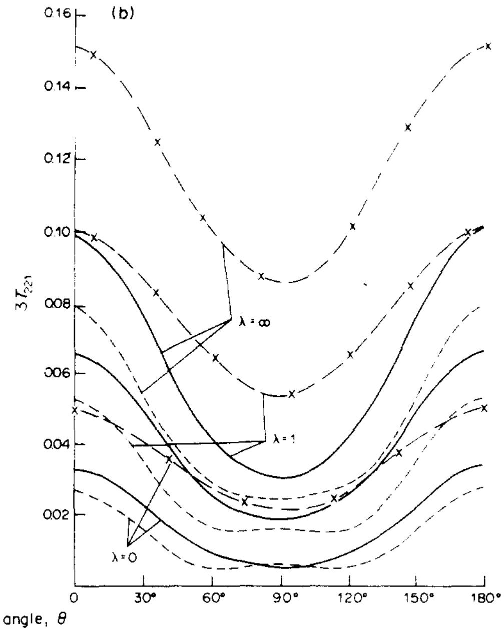

Figure 12b. Dimensionless third-order torque, T_{221}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity $w_2 e_2$: ---, case I; ——, case II; -- × --, case III.

component of T_2 has the same sign for all values of λ and θ plotted. The only non-zero component F_1, of the force is obtained by multiplying the r.h.s. of [62b] by w_3/U_3.

6. APPLICATION TO SLENDER BODIES

The analysis of section 3 can be applied to slender bodies or other bodies for which the solution to [20] is known only approximately. For a slender body, whose half-length is l, appropriate forms of [20] and [27] are

$$\alpha_i + a_{ij}x_j + \alpha_{ijk}x_k x_j + \cdots = \int_{-l}^{l} f_j(\xi_1, \xi_2, \xi_3)(2\delta_{ij}R^{-1} - R_{,ij})\, \mathrm{d}\rho, \tag{81}$$

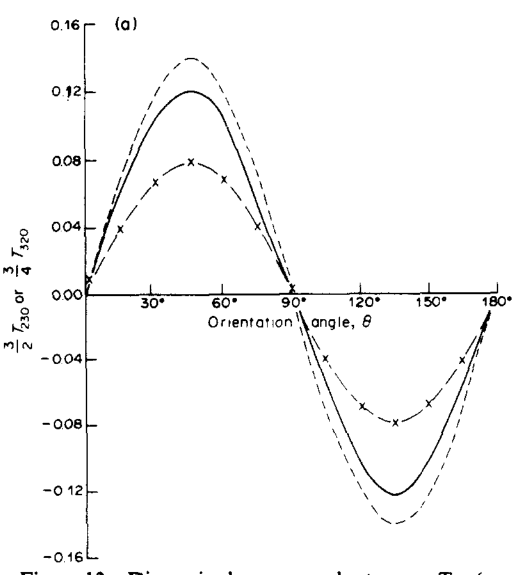

Figure 13a. Dimensionless zero-order torque, T_{230} (or T_{320}) as a function of the orientation angle θ for an ellipsoid rotating with angular velocity $w_2 e_2$ (or $w_3 e_3$): ---, case I; ——, case II; -- × --, case III.

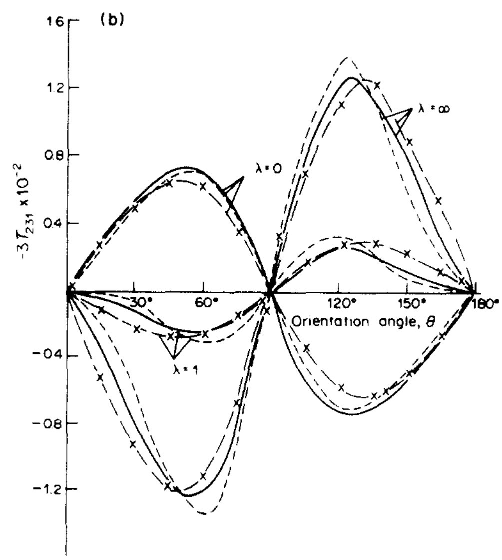

Figure 13b. Dimensionless third-zero torque, T_{231}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity $w_2 e_2$: ---, case I; ——, case II; -- × --, case III.

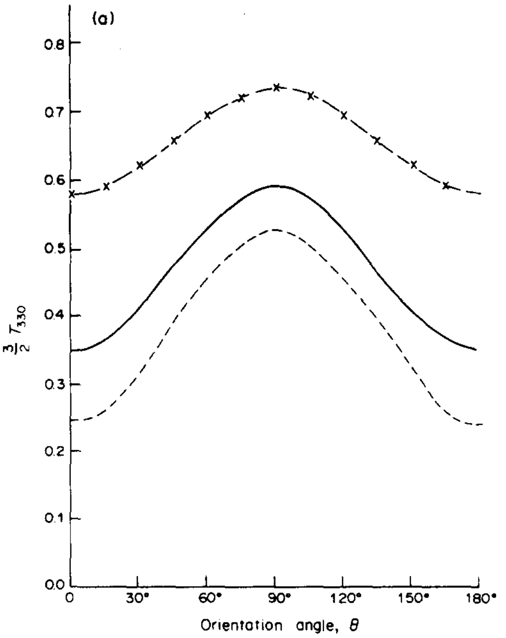

Figure 14a. Dimensionless zero-order torque, T_{330}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity w_3e_3: ---, case I; ——, case II; -- x --, case III.

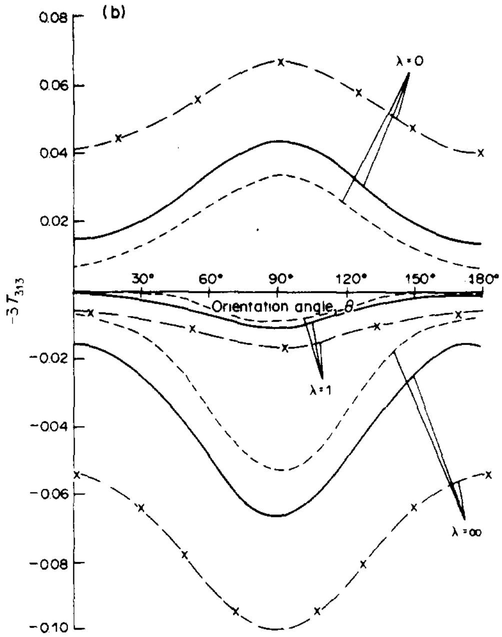

Figure 14b. Dimensionless third-order torque, T_{331}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity w_3e_3: ---, case I; ——, case II; -- x --, case III.

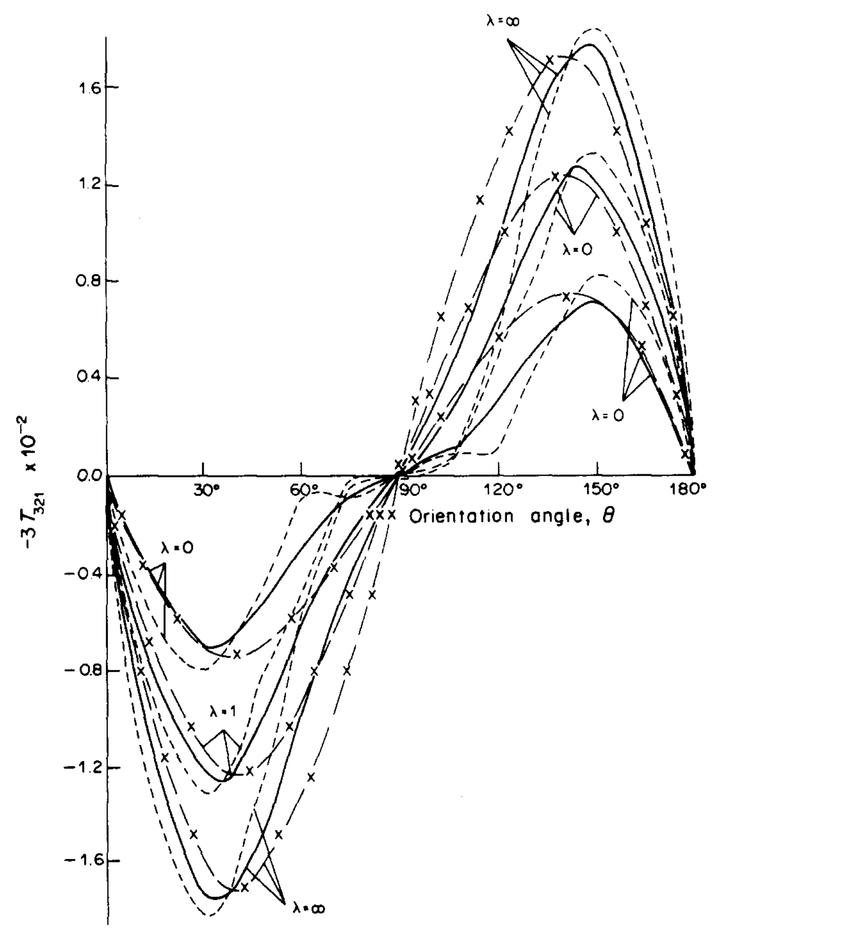

Figure 15. Dimensionless third-order torque, T_{321}, as a function of the orientation angle θ for an ellipsoid rotating with angular velocity w_3e_3: ---, case I; ——, case II; -- x --, case III.

$$A_j = \frac{1}{8\pi}\int_{-l}^{l} f_j \, d\rho, \quad A_{jk} = -\frac{1}{8\pi}\int_{-l}^{l} f_j \xi_k \, d\rho,$$

$$A_{jkl} = \frac{1}{16\pi}\int_{-l}^{l} f_j \xi_k \xi_l \, d\rho \text{ etc.,} \qquad [82]$$

where ξ_1, ξ_2 and ξ_3 are coordinates of points on the body axis and ρ is the distance measured along the centreline from the centre, O, of the body. It is worth noting however, that [82] can not be exactly satisfied up to and higher than the third order in the slenderness ratio, ϵ, without requiring the distribution of higher-order singularities (e.g. potential dipoles, doublets etc.) on the body axis; ϵ is defined as

$$\epsilon = \ln\left(\frac{21}{R_0}\right)^{-1},$$

where R_0 is the maximum effective radius of the body.

For illustration purposes, we now calculate the force and torque experienced by an arbitrarily-oriented slender circular cylindrical body when the body is moving normal to the interface with speed U_3. The coordinate system, (x_1, x_2, x_3) is chosen such that the body axis lies entirely in the x_2–x_3 plane (i.e. $\xi_1 \equiv 0$) and makes an angle θ with the x_3-axis, as shown in figure 16. In this system, the body centre, O, has coordinates $(0, 0, d)$. We choose l and U_3 as the characteristic length and speed, respectively, with respect to which other lengths and velocities are non-dimensionalized. Also $K \equiv l/d$.

For this slender body, an approximate solution of [81] is (Batchelor 1970)

$$f_1 = 0, \qquad [83]$$

$$f_2 = -2\cos\theta\sin\theta\,\epsilon\left[\!\left[1 - \tfrac{1}{2}\epsilon\left\{\ln\left[1 - \left(\frac{\rho}{l}\right)^2\right] + 3\right\} + O(\epsilon^2)\right]\!\right] \qquad [84]$$

and

$$f_3 = -2(1 + \cos^2\theta)\epsilon\left[\!\left[1 - \tfrac{1}{2}\epsilon\left\{\ln\left[1 - \left(\frac{\rho}{l}\right)^2\right] + \frac{3\sin^2\theta - 1}{2(\sin^2\theta - 1)}\right\} + O(\epsilon^2)\right]\!\right]. \qquad [85]$$

Also,

$$A_2 = -\tfrac{1}{2}\cos\theta\sin\theta\,\epsilon\,[1 - \epsilon(\ln 2 + \tfrac{1}{2}) + O(\epsilon^2)], \qquad [86]$$

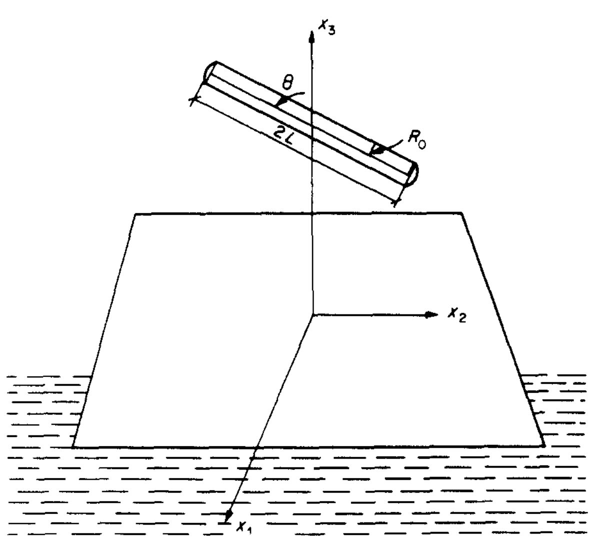

Figure 16. A sketch of the coordinate system and the position of the slender circular cylindrical body.

Table 3. Tabulation of the dimensionless normal force, $F_2/\pi\mu^1 l\epsilon U_3$, drag force, $F_3/\pi\mu^1 l\epsilon U_3$ and torque $T_1/\pi\mu^1 l^2\epsilon^2 U_3$ as a function of θ for a slender circular cylinder translating normal to an interface

θ	$-F_2/\pi\mu^1 l\epsilon U_3$			$-F_3/\pi\mu^1 l\epsilon U_3$			$-T_1/\pi\mu^1 l^2\epsilon^2 U_3$			
(deg)	$\lambda = 0$	$\lambda = 1$	$\lambda = \infty$	$\lambda = 0$	$\lambda = 1$	$\lambda = \infty$	$\lambda = 0$	$\lambda = 1$	$\lambda = \infty$	
0° (180°)	0.0000	0.000000	0.000	15.266	15.463	15.660	0.0000	0.0000	0.0000	$K = 0.5$
15° (165°)	2.4649 (−)	2.6340 (−)	2.8032 (−)	15.992	16.206	16.42	0.2641 (−)	0.32539 (−)	0.38671 (−)	
30° (150°)	4.3116 (−)	4.5940 (−)	4.8764 (−)	17.988	18.256	18.525	0.50171 (−)	0.59685 (−)	0.69198 (−)	
45° (135°)	5.0454 (−)	5.3548 (−)	5.6642 (−)	20.754	21.105	21.456	0.64928 (−)	0.7464 (−)	0.83401 (−)	
60° (120°)	4.4272 (−)	4.6807 (−)	4.9342 (−)	23.564	24.009	24.454	0.62287 (−)	0.68772 (−)	0.75256 (−)	
75° (105°)	2.5805 (−)	2.7207 (−)	2.8610 (−)	25.649	26.170	26.691	0.38522 (−)	0.41626 (−)	0.44730 (−)	
90°	0.0000	0.0000	0.0000	26.418	26.969	27.519	0.0000	0.0000	0.0000 (−)	
0° (180°)	0.0000	0.0000	0.0000	16.038	16.054	16.369	0.0000	0.0000	0.0000	$K = 0.8$
15° (165°)	2.4833 (−)	2.7539 (−)	3.0246 (−)	16.507	16.851	17.195	0.6760 (−)	0.8299 (−)	0.9899 (−)	
30° (150°)	4.3688 (−)	4.8207 (−)	5.2725 (−)	18.632	19.062	19.491	1.2844 (−)	1.5279 (−)	1.7715 (−)	
45° (135°)	5.1540 (−)	5.6465 (−)	6.1416 (−)	21.597	22.159	22.721	1.6622 (−)	1.8986 (−)	2.1351 (−)	
60° (120°)	4.5538 (−)	4.9594 (−)	5.3650 (−)	24.632	25.344	26.057	1.5946 (−)	1.7606 (−)	1.9266 (−)	
75° (105°)	2.6682 (−)	2.8926 (−)	3.1170 (−)	26.900	27.734	28.567	0.9862 (−)	1.0656 (−)	1.1451 (−)	
90°	0.0000	0.0000	0.000	27.793	28.620	29.501	0.0000	0.000	0.0000	
0° (180°)	0.0000	0.0000	0.0000	16.038	16.428	16.818	0.0000	0.0000	0.0000	$K = (1.01)^{-1}$
15° (165°)	2.4949 (−)	2.8299 (−)	3.1648 (−)	16.834	17.260	17.685	1.0354 (−)	1.2759 (−)	1.5164 (−)	
30° (150°)	4.4051 (−)	4.9643 (−)	5.5235 (−)	19.041	19.572	20.104	1.9673 (−)	2.3403	2.7134 (−)	
45° (135°)	5.2187 (−)	5.8314 (−)	6.4441 (−)	22.131	22.827	23.523	2.3523 (−)	2.9081 (−)	3.2703	
60° (120°)	4.6340 (−)	5.1360 (−)	5.6380 (−)	25.310	26.191	27.073	2.4424 (−)	2.6967 (−)	2.9509 (−)	
75° (105°)	2.7238 (−)	3.0015 (−)	3.2793 (−)	27.692	28.724	29.756	1.5105 (−)	1.6322 (−)	1.7539 (−)	
90°	0.0000	0.0000	0.0000	28.576	29.666	30.756	0.0000	0.0000	0.000	

$$A_3 = -\tfrac{1}{2}(1+\sin^2\theta)\,\epsilon\left\{1-\epsilon\left[\ln 2 - 1 + \frac{3\sin^2\theta - 1}{2(\sin^2\theta+1)}\right] + O(\epsilon^2)\right\}, \qquad [87]$$

$$C_{33} = A_3, \qquad [88]$$

$$C_{23} = C_{32} = A_2 \qquad [89]$$

and

$$C_{22} = -\tfrac{1}{2}(1+\cos^2\theta)\epsilon\left\{1-\epsilon\left[\ln 2 - 1 + \frac{3\cos^2\theta - 1}{2(1+\cos^2\theta)}\right] + O(\epsilon^2)\right\}. \qquad [90]$$

The non-zero components of the dimensional force, F_j, and torque, T_j, which the body experiences may be deduced from [42] and [43] to be

$$F_2 = 8\pi\mu^1 U_3 l\,[A_2 + K(A_3P_{33}C_{32} + A_2P_{22}C_{22}) + O(\epsilon^3, K^2)], \qquad [91]$$

$$F_3 = 8\pi\mu^1 U_3 l\,[A_3 + K(A_3P_{33}C_{33} + A_2P_{22}C_{23}) + O(\epsilon^3, K^2)] \qquad [92]$$

and

$$T_1 = 8\pi\mu U_3 l^3 K^2\{A_3[\bar{P}_{322}(D_{2223} - D_{2332}) + \bar{P}_{333}(D_{3323} - D_{3332})] + A_2[\bar{P}_{232}(D_{2323} - D_{2332}) + \bar{P}_{223}(D_{3223} - D_{3232}) + O(\epsilon^3, K^3)]\}. \qquad [93]$$

In [93],

$$(D_{2223} - D_{2232}) - 2(D_{3323} - D_{3332}) = \cos\theta\sin\theta\,\epsilon\,[1 + \epsilon\,(\tfrac{35}{6} - 2\ln 2)], \qquad [94]$$

$$D_{2323} - D_{2232} = -\tfrac{1}{3}\cos^2\theta\,\epsilon\,[1 + \epsilon\,(\tfrac{11}{6} - 2\ln 2)] \qquad [95]$$

and

$$D_{3223} - D_{3232} = \tfrac{1}{3}\sin^2\theta\,\epsilon\,[1 + \epsilon\,(\tfrac{11}{6} - 2\ln 2)]. \qquad [96]$$

It should be pointed out that the non-zero components of A_j, C_{ij} and D_{ijkl} $(i,j,k,l = 2,3)$ have leading terms of order ϵ. Consequently, remainder terms of order $K^n (n = 1, 2, \ldots)$ in [91]–[93] actually have leading terms of order ϵ^n as factors. Therefore, [91]–[93] are valid to the stated order in ϵ alone, even when $K(\equiv l/d)$ is of the order of unity.

The computed values of $-F_2/(\pi\mu^1 l\epsilon U_3)$, $-F_3/(\pi\mu^1 l\epsilon U_3)$ and $-T_1/(\pi\mu^1 l^2\epsilon^2 U_3)$ are tabulated as functions of θ for $\lambda = 0$, $\lambda = 1$ and $\lambda = \infty$ and for $K = 0.5$, $K = 0.8$ and $K = (1.01)^{-1}$ in table 3. All the tabulations are for $\epsilon = 0.1887$. Agreement with the corresponding results of Yang & Leal (1983) is excellent.

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